DEHN FUNCTIONS
EXERCISES FOR DAY 1 (MONDAY, JUNE 15) AND REFERENCES

Exercises.

(1) (Get a feel for $\simeq$-equivalence) Order these functions with respect to $\preceq$. Determine which, if any, are $\simeq$-equivalent.

$$n^2 3^n, \quad 4^{n^2}, \quad 2^{2^n}, \quad n^2(\log n)\, 2^n, \quad n 2^{n^2+2}, \quad n^{10} + 10^n, \quad \sqrt{n}\, 3^{\sqrt{n}}, \quad f(n) \text{ (defined as below)}$$

Define $f(n) = \exp^n(e)$, where $\exp(n) = e^n$.

(2) (This example shows why we need the linear term in the definition of $\simeq$-equivalence.) Show that the Dehn function of $\langle a \mid \emptyset \rangle$ is $\delta(n) \equiv 0$, and the Dehn function of $\langle a, b \mid b \rangle$ is $\delta(n) = n$

This is Exercise 1.3.1 in Bridson's *The geometry of the word problem.*

(3) Show that $\delta_{\mathbb{Z}^m}(n) \simeq n^2$ for $m > 1$.

(4) The *Heisenberg group* is the group with presentation

$$H = \langle a, b, c \ \mid \ [a, b] = c,\ [a, c] = [b, c] = 1 \rangle.$$

Use the fact that every element can be expressed in a unique way as a word of the form $a^p b^q c^r$ for some integers p, q, r to show that $\delta_H(n) \preceq n^3$.

This is Exercise 2.14 in Riley's *What is a Dehn function?*

(5) (a) Prove that if P_1 and P_2 are two different finite presentations for a group G, then $\delta_{P_1}(n) \simeq \delta_{P_2}(n)$. (Use the fact that any two presentations of G are related by a sequence of *Tietze moves.*)

(b) If you are familiar with quasi-isometries, sketch a proof that the Dehn function of a group is a quasi-isometry invariant. (The idea is straightforward, but the details get technical.)

(6) Show that if $G = \langle S \mid R \rangle$ is finitely presented, and $w \in F(S)$ with $w =_G 1$ and $|w| = n$, then there is an expression

$$\prod_{i=1}^{m} u_i r_{j_i} u_i^{-1}$$

with $|u_i| \leq |w| + C\delta_G(n)$, where C is the maximum of the lengths of the relators in R.

References.

(1) Martin Bridson, *The geometry of the word problem.* Invitations to geometry and topology, 2991, Oxf. Grad. Texts Math., 7, Oxford Univ. Press, Oxford, 2002.

(2) Steve Gersten, *Isoperimetric and isodiametric functions of finite presentations.* Geometric group theory, Vol. 1 (Sussex, 1991), 7996, London Math. Soc. Lecture Note Ser., 181, Cambridge Univ. Press, Cambridge, 1993.

(3) Tim Riley, *Filling functions.* Chapter in *The geometry of the word problem for finitely generated groups.* Papers from the Advanced Course held in Barcelona, July 5–15, 2005. Advanced Courses in Mathematics. CRM Barcelona. Birkhuser Verlag, Basel, 2007.

(4) Tim Riley, *What is a Dehn function?* A chapter for a forthcoming book Office hours with a geometric group theorist, edited by Matt Clay and Dan Margalit, Princeton University Press.
Available at `http://www.math.cornell.edu/tim.riley/articles.html`

(5) Mark Sapir, *Asymptotic invariants, complexity of groups and related problems.* Bull. Math. Sci. 1 (2011), no. 2, 277–364.

DEHN FUNCTIONS
EXERCISES FOR DAY 2 (TUESDAY, JUNE 16)

(1) Finish the proof from the lecture, that hyperbolic groups have a Dehn presentation.

(2) The group $BS(1,2) = \langle a, t \mid tat^{-1} = a^2 \rangle$ is a *Baumslag-Solitar* group. Sketch a picture of the universal cover of its presentation complex. This will be useful tomorrow.

(3) Show that $\mathbb{Z}^n$ has a synchronous 2-fellow traveling combing.

(4) (Asynchronous combings) Define a reparametrization to be a function $\rho : \mathbb{N} \to \mathbb{N}$ with $\rho(0) = 0$ and $\rho(n) \in \{\rho(n-1), \rho(n-1)+1\}$ for all $n \geq 1$.

A normal form σ for G with generating set S *asynchronously k-fellow travels* if for all $g, h \in G$ with $d_G(g,h) = 1$, there exist reparametrizations ρ, ρ' such that $d_S(\sigma_g(\rho(t), \sigma_h(\rho(t))) \leq k$. Now (G,S) is *asynchronously combable* if there exists an asynchronous k-fellow traveling normal form for G. As in the lecture, define the length function by

$$L(n) = \max\{\ell(\sigma_g) \mid d_G(g,e) = 1\rangle$$

Show that if (G,S) has an asynchronous k-fellow traveling combing then

$$\delta_G(n) \preceq n\, L(n)$$

by modifying the proof from the lecture.

The Baumslag-Solitar group $BS(1,2) = \langle a, t \mid tat^{-1} = a^2 \rangle$ is asynchronously combable with $L(n) \simeq 2^n$. See Riley's *Filling Functions* for a description of the normal form that works.

(5) (Exercise 3.4.1 from Bridson's *The geometry of the word problem*)

(a) A subgroup H of a group G is a *retract* if there is a homomorphism $G \to H$ whose restriction to H is the identity. Show that if H is a retract of a finitely presented group G, then H is finitely presented, and $\delta_H(n) \preceq \delta_G(n)$.
(Hint: First note that H is finitely generated. Show that a generating set for H can be extended to one for G, and use this to produce a finite presentation for G.)

(b) Let G_1 and G_2 be infinite, finitely presented groups. Show that

$$\delta_{G_1 \times G_2}(n) \simeq \max\{n^2, \delta_{G_1}(n), \delta_{G_2}(n)\}$$

and

$$\delta_{G_1 * G_2}(n) \simeq \max\{\delta_{G_1}(n), \delta_{G_2}(n)\}.$$

(6) Show that the slim triangles condition is equivalent to the thin triangles condition for hyperbolicity. Specifically, show that any η-slim triangle is 4η thin.

(This requires quite a careful use of the triangle inequality. If you get stuck, look at these MSRI notes from 25 years ago! –
`https://www.i2m.univ-amu.fr/~short/Papers/MSRInotes2004.pdf`)

DEHN FUNCTIONS
EXERCISES FOR DAY 3 (WEDNESDAY, JUNE 17)

(1) Show that if H is an HNN extension of G defined by an injective homomorphism $\phi : G \to B < G$ then there is an exponential upper bound on the Dehn function of G.

(2) Finish the proof that the Dehn function of $BS(1,2)$ is 2^n.

(3) $G = \langle a, ts \mid tat^{-1} = a^2, sts^{-1} = t^2 \rangle$. Show that the Dehn function is 2^{2^n}.

(4) Define analogous groups (to the one define in the previous problem) with Dehn functions equal to any tower of exponential functions.

(5) Show that the group presented by $G = \langle a, t, s \mid tat^{-1} = a^2, sas^{-1} = t \rangle$ (Gersten's group) has Dehn function growing faster than any tower of exponentials.

DEHN FUNCTIONS
EXERCISES FOR DAY 4 (THURSDAY, JUNE 18)

(1) Complete the details of the proof that the Dehn function of the Heisenberg group is n^3.

(2) Complete the details of the proof that the Dehn function of the lattice in Sol is exponential.

(3) Given a pair of groups $\Gamma < G$ with word metrics d_H and d_G, the *distortion* of Γ in G is

$$\mathrm{disto}_\Gamma^G(n) = \max\{d_\Gamma(e,g) \mid g \in \Gamma; d_G(e,h) \leq n\}$$

(a) If $H = \langle a,b,c \mid [a,b] = c,\ [a,c] = [b,c] = 1\rangle$ is the Heisenberg group, what is $\mathrm{disto}_{\langle c\rangle}^H(n)$?

(b) If $G = BS(1,2) = \langle a,t \mid tat^{-1} = a^2\rangle$ what is $\mathrm{disto}_{\langle a\rangle}^G(n)$?

(c) Define $\phi : F(a,b,c) \to F(a,b,c)$ by $a \mapsto ab$, $b \mapsto bc$, and $c \mapsto c\rangle$. Now let G_t be the HNN extension

$$G_t = \langle a,b,c,t \mid txt^{-1} = \phi(x) \text{ for } x = a,b,c\rangle$$

(i) What is the distortion of $F(a,b,c)$ in G_t?

(ii) Define G_s analogously to G_t, with s taking the place of t. What can you say about the Dehn function of the group $G_t *_{F(a,b,c)} G_s$? (This group is sometimes called the *double* of G_t over $F(a,b,c)$.

(4) *CAT(0) groups* are groups which act properly discontinuously and cocompactly by isometries on a *CAT(0) space*. To find combings, one only needs an important consequence of the CAT(0) condition, namely the fact that the metric on a CAT(0) space X is *convex*. This means that given a pair of geodesics $\alpha, \beta : [0,1] \to X$ parametrized proportional to arclength, one has

$$d_X(\alpha(t), \beta(t)) \leq t d_X(\alpha(0), \beta(0)) + (1-t) d_X(\alpha(1), \beta(1))$$

Show that if X has a convex metric, and G acts on X properly discontinuously and cocompactly by isometries, then G is synchronously combable, with linear length function.

Thus any such G satisfies a quadratic isoperimetric inequality.

DEHN FUNCTIONS
EXERCISES FOR DAY 5 (FRIDAY, JUNE 19)

(1) Let G be a finitely presented group. Show that the word problem in G is solvable if and only if the Dehn function of G is recursive.

(Take recursive to mean that there is an algorithm which, given n, computes $f(n)$ in a finite amount of time.)

(2) Finish the proof of Groft's theorem. Let X be the universal cover of a $K(G,1)$ with finite $(k+1)$-skeleton.

(a) Let $k \geq 2$. Show that given a k-sphere $\sigma : S^k \to X^{(k)}$, and a filling of the corresponding homology class $\hat{\sigma}$ by a $(k+1)$-chain β, there is an extension $\tilde{\sigma} : D^{k+1} \to X^{k+1}$ such that $\mathrm{Vol}(\tilde{\sigma}) = \mathrm{mass}(\beta)$.

This shows that for a given sphere, you can do no better by using chains to fill than by using just disks.
(This was worked out in the problem session.)

(b) Let $k \geq 3$. Show that given any cycle $\alpha \in Z_k(X, \mathbb{Z})$, there exists a k-sphere $\sigma : S^k \to X^{(k)}$ such that

$$\mathrm{Vol}(\sigma) = \mathrm{mass}(\alpha) \text{ and } \mathrm{FVol}(\sigma) = \mathrm{FV}(\alpha)$$

(c) Explain why the above statements fail when k is lower than the threshold mentioned.

(d) Use the statements in (1) and (2) to finish the proof of Groft's Theorem.