

The membrane shell model in nonlinear elasticity: A variational asymptotic derivation

Hervé LE DRET¹ and Annie RAOULT²

Abstract — We consider a shell-like three-dimensional nonlinearly hyperelastic body and we let its thickness go to zero. We show, under appropriate hypotheses on the applied loads, that the deformations that minimize the total energy weakly converge in a Sobolev space toward deformations that minimize a nonlinear shell membrane energy. The nonlinear shell membrane energy is obtained by computing the Γ -limit of the sequence of three-dimensional energies.

¹ Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie, 75252 Paris Cedex 05, France

² Laboratoire de Modélisation et Calcul, Université Joseph Fourier, BP 53, 38041 Grenoble Cedex 9, France

1. Introduction

The purpose of this article is to derive nonlinear membrane shell models from genuine three-dimensional nonlinear elasticity by means of a rigorous convergence result. It is a sequel to a previous article concerned with planar membranes, see Le Dret and Raoult [1995].

J.C. Simo's profound interest in the large deformation theory of thin structures is at the origin of numerous works on the derivation, analysis and approximation of models for shells and rods that respect the fundamental requirement of continuum mechanics, frame-indifference. These models, see Simo and Vu-Quoc [1988,1991], Simo and Fox [1989,1992], Simo, Fox and Rifai [1990a, 1990b], Simo and Tarnow [1994] for instance, rely on a kinematic assumption on the possible deformed configurations of the body. In this framework, shells are one-director Cosserat structures. The shell model constructed in Simo and Fox [1989] is fully nonlinear, frame-indifferent and couples membrane, bending and shearing effects together. Recent results on the numerical approximation and on the construction of such a model are given in Carrive-Bédouani, Le Tallec and Mouro [1995], Carrive-Bédouani [1995]. Important contributions to the formulation of classical nonlinear shell theory using the Cosserat hypothesis as well as thorough and comprehensive analysis of these models can be found in *e.g.* Ericksen and Truesdell [1958], Naghdi [1972], Green and Naghdi [1974], Antman [1976a, 1976b, 1995].

A complementary approach to thin elastic structures theory is that of formal asymptotic expansions in powers of the thickness pioneered by Friedrichs and Dressler [1961] and Goldenveizer [1963] and later recast in a modern functional framework by Ciarlet and Destuynder [1979a, 1979b], Ciarlet [1980] and in the case of shells Destuynder [1980]. This approach also led to numerous developments, see Ciarlet [1990] for a bibliography. A one year stay of the second author at the Division of Applied Mechanics at Stanford University provided the opportunity to try and bridge the two approaches. More specifically, the goal was to investigate whether the asymptotic expansion method applied to thin bodies could lead to frame-indifferent limit models.

This objective is attained in Fox, Raoult and Simo [1993] where, for simplicity, the case of plate-like—rather than shell-like—bodies is treated and where the nonlinear material is the Saint Venant-Kirchhoff material. It is shown that a hierarchy of two-dimensional models can be derived from the nonlinear system of three-dimensional elasticity by a formal asymptotic expansion. The type of limit model thus obtained depends on the order of magnitude of the external loads. The first two models in the hierarchy are a nonlinear membrane plate model and a nonlinear inextensional bending model for smaller loads. Both models are quasilinear and frame-indifferent. The membrane model is also obtained by Karwowski [1993]. By lowering again the order of magnitude of the loads, one recovers semilinear plate models that had been previously derived by Ciarlet and Destuynder [1979b], Ciarlet [1980] for the von Kármán equations and Raoult [1988] in the dynamical case by analogous formal expansions. Note that these models are no longer frame-indifferent.

Although establishing the grounds for an asymptotic justification of invariant plate models, the method of Fox, Raoult and Simo [1993] is purely formal. It was the purpose of the work by Le Dret and Raoult [1993, 1995] to provide a rigorous proof of convergence to a nonlinear membrane model. In this work, as in Fox, Raoult and Simo [1993], only plate-like bodies are considered. The assumptions on the external loads are those of Fox, Raoult and Simo [1993], but the results are obtained for a general hyperelastic material. The main mathematical tool is Γ -convergence theory, a systematic way of analysing the convergence of minimizers of a sequence of problems of the Calculus of Variations. Ideas from Γ -convergence theory had been previously introduced in the context of lower-dimensional theories in nonlinear elasticity by Acerbi, Buttazzo and Percivale [1991] for nonlinearly elastic strings. Their method was extended to nonlinear planar membranes in a preprint by Percivale [1991] recently drawn to the authors' attention. Note that in the case of strings the limit model is one-dimensional and thus convexity arguments can be used that are not sufficient in the two-dimensional case. Let us briefly recall the limit membrane model obtained in Le Dret and Raoult [1993, 1995], see also Percivale [1991]. Starting from a three-dimensional stored energy function W defined on three-dimensional deformation gradients, *i.e.*, 3×3 matrices, the limit membrane energy density, which is defined on membrane deformation gradients, *i.e.*, 3×2 matrices, is constructed in two steps. First W is minimized with respect to the third column of its matrix argument, then the resulting function is quasiconvexified. It is worth mentioning that, except in some very special cases, this quasiconvexification step cannot be skipped. In the case of the Saint Venant-Kirchhoff density, the quasiconvexification is carried out explicitly in Le Dret and Raoult [1995]. A quite surprising consequence of this calculation is that the formal limit membrane energy of Fox, Raoult and Simo [1993] only coincides with the rigorous limit energy on a compact subset of the set of 3×2 matrices.

The present article extends the analysis of Le Dret and Raoult [1995] to the case of shells. Let us mention other recent works in the field of asymptotic justification and analysis of lower-dimensional linear or nonlinear shell models: Sanchez-Palencia [1989a, 1989b, 1990], Ciarlet and Lods [1994], Ciarlet, Lods and Miara [1994], Miara [1994].

An overview of the article is as follows. Section 2 is devoted to introducing the geometrical notation for a shell with mid-surface \tilde{S} and thickness 2ε in its reference configuration $\tilde{\Omega}_\varepsilon$. We assume that the shell is made of a hyperelastic homogeneous material with stored energy function W . In Section 3, we state the equilibrium problem for the shell as an energy minimization problem over a set of admissible deformations included in the Sobolev space $W^{1,p}(\tilde{\Omega}_\varepsilon; \mathbb{R}^3)$. To study the asymptotic behavior of the corresponding energy minimizers when $\varepsilon \rightarrow 0$, we define an equivalent minimization problem set on a straight cylindrical domain $\Omega = \omega \times]-1, 1[$ of \mathbb{R}^3 independent of ε . This is achieved by transporting the deformations and the external loads through a chart and rescaling them. As opposed to the planar case, the geometry of the shell intervenes in the expression of the rescaled hyperelastic energies through the Jacobian matrix of the change of coordinates. This

matrix depends on ε and appears notably inside the argument of the stored energy density.

In Section 4, we give our first convergence result expressed in terms of rescaled displacements. We determine the Γ -limit of the sequence of rescaled energies. The construction of the limit energy density extends that of the planar case. Note however that the nontrivial geometry of the shell causes the limit energy to depend on the point $x \in \omega$ even for a homogeneous three-dimensional material.

In Section 5, we translate the Γ -convergence result of Section 4 in terms of deformations and show that the minimizing deformations weakly converge in $W^{1,p}(\Omega; \mathbb{R}^3)$ towards deformations that minimize a limit nonlinear shell energy. The limit deformations depend on two space variables only (they are identified with functions defined on the transported mid-surface ω). The limit elastic energy of a deformation $\bar{\varphi}$ depends on its first derivatives and thus does not incorporate bending effects associated with curvature nor shear effects. Consequently, it is a membrane energy. Let us emphasize the fact that, contrarily to methods relying on Cosserat assumptions, our analysis provides an exact formula for deriving the limit energy, hence the membrane constitutive law, from the three-dimensional energy W . Furthermore, we give an intrinsic formulation of the membrane minimization problem and an intrinsic expression of the nonlinear membrane energy by transporting the obtained result back on the reference surface \tilde{S} . The stored energy depends on a deformation $\tilde{\varphi}$ defined on \tilde{S} only through its gradient (for a definition of this gradient, see Section 5). It depends on the current point of \tilde{S} , but only through the normal vector to \tilde{S} at this point.

In Section 6, we study how the limit membrane shell energy inherits the invariance properties of the three-dimensional energy. In particular, it is shown that frame-indifference is preserved and that the shell energy depends on the deformation only through the deformed metric. Moreover, if W has a global zero minimum at $F = I$, the corresponding membrane shell energy is zero for compressive states. Isotropy is also preserved. In this case, the membrane stored energy does no longer depend on the normal vector to the reference surface \tilde{S} and depends on the deformation only through the principal stretches.

The second author is indebted to J.C. Simo for inviting her to spend a sabbatical year at the Division of Applied Mechanics at Stanford University in 1990–1991. Her work owes much to J.C. Simo's brilliance and enthusiasm.

2. Geometrical preliminaries

The summation convention is assumed throughout this article, unless otherwise specified. Greek indices take their values in the set $\{1, 2\}$ and Latin indices take their values in the set $\{1, 2, 3\}$. Let (e_1, e_2, e_3) be the canonical orthonormal basis of the Euclidean space \mathbb{R}^3 . The norm of a vector of \mathbb{R}^3 will be denoted by $\|u\|$, the scalar product of two vectors of \mathbb{R}^3 by $u \cdot v$ and their vector product by $u \wedge v$. In the sequel, we will identify \mathbb{R}^2 with the plane spanned by the vectors e_1 and e_2 . Accordingly and depending on the context, x will denote a generic point of \mathbb{R}^2 or \mathbb{R}^3 . Let M_3

be the space of real 3×3 matrices endowed with the usual Euclidean norm $\|F\| = \sqrt{\text{tr}(F^T F)}$. For any $z_i \in \mathbb{R}^3$, $i = 1, 2, 3$, we note $(z_1|z_2|z_3)$ the matrix whose i -th column consists of the components of z_i in the canonical basis.

We assume that the midsurface \tilde{S} is a bounded, two-dimensional, C^2 -submanifold of \mathbb{R}^3 , which, for simplicity, admits an atlas consisting of one chart only. Let ψ be this chart. It is thus a C^2 -mapping from a bounded, open set $\omega \subset \mathbb{R}^2$ into \mathbb{R}^3 which is a global diffeomorphism between ω and \tilde{S} . We assume that ω has a Lipschitz boundary and that ψ admits an extension to $\bar{\omega}$ into a $C^2(\bar{\omega}; \mathbb{R}^3)$ -function.

Let $a_\alpha(x) = \frac{\partial \psi}{\partial x_\alpha}(x)$ be the covariant basis vectors of the tangent plane $T_{\psi(x)}\tilde{S}$, associated with the chart ψ . These vectors are linearly independent on ω . We assume furthermore that there exists $\delta > 0$ such that

$$\|a_1(x) \wedge a_2(x)\| \geq \delta \text{ on } \bar{\omega}. \quad (1)$$

We then define $a_3(x) = \frac{a_1(x) \wedge a_2(x)}{\|a_1(x) \wedge a_2(x)\|} \in C^1(\bar{\omega}; S^2)$, which is a unit normal vector to $T_{\psi(x)}\tilde{S}$. If no confusion may arise from it, we will write $a_3(\tilde{x})$ for $a_3(x)$ at point $\tilde{x} = \psi(x)$, since it is important to remember that a_3 is actually chart-independent (modulo multiplication by -1). The contravariant basis vectors are defined by the relations $a^\alpha(x) \in T_{\psi(x)}\tilde{S}$, $a^\alpha(x) \cdot a_\beta(x) = \delta_{\alpha\beta}$ and $a^3(x) = a_3(x)$.

Using this notation, we let $A(x) = (a_1(x)|a_2(x)|a_3(x))$. This matrix is everywhere nonsingular on $\bar{\omega}$ and its inverse is given by $A^{-1}(x) = (a^1(x)|a^2(x)|a^3(x))^T$. Note that due to our choice of unit normal vector, $\det A(x) = \|a_1(x) \wedge a_2(x)\| \geq \delta > 0$ on $\bar{\omega}$. Thus, there exists a constant C such that

$$\forall x \in \bar{\omega}, \quad \|A^{-1}(x)\| \leq C. \quad (2)$$

The function $\det A(x)$ also satisfies

$$\det A(x) = \|\text{cof } A(x)e_3\| = \sqrt{a(x)} \quad (3)$$

where a is the determinant of the metric on \tilde{S} expressed in the chart ψ .

For $\varepsilon > 0$, we consider the set $\tilde{\Omega}_\varepsilon$ defined by

$$\tilde{\Omega}_\varepsilon = \{\tilde{y} \in \mathbb{R}^3; \exists \tilde{x} \in \tilde{S}, \tilde{y} = \tilde{x} + \eta a_3(\tilde{x}) \text{ with } |\eta| < \varepsilon\}. \quad (4)$$

This set is the reference configuration of a shell of thickness 2ε . Due to our regularity hypothesis on \tilde{S} there exists a C^1 -orthogonal projection mapping $\tilde{\Pi} : \tilde{\Omega}_\varepsilon \rightarrow \tilde{S}$ if ε is small enough, which will be understood thereafter. Any \tilde{y} in $\tilde{\Omega}_\varepsilon$ can be uniquely decomposed as $\tilde{y} = \tilde{\Pi}(\tilde{y}) + [(\tilde{y} - \tilde{\Pi}(\tilde{y})) \cdot a_3(\tilde{\Pi}(\tilde{y}))]a_3(\tilde{\Pi}(\tilde{y}))$. With this notation, $(x_1, x_2) = \psi^{-1}(\tilde{\Pi}(\tilde{y}))$ and $x_3 = (\tilde{y} - \tilde{\Pi}(\tilde{y})) \cdot a_3(\tilde{\Pi}(\tilde{y}))$ define the natural curvilinear coordinate system in $\tilde{\Omega}_\varepsilon$ that is associated with the chart ψ of the midsurface. If

$$\Omega_\varepsilon = \{x \in \mathbb{R}^3; (x_1, x_2) \in \omega, |x_3| < \varepsilon\}, \quad (5)$$

then the C^1 -diffeomorphism $\Psi : \Omega_\varepsilon \rightarrow \tilde{\Omega}_\varepsilon$ defined by

$$\Psi(x) = \psi(x_1, x_2) + x_3 a_3(x_1, x_2) \quad (6)$$

is the inverse of this change of coordinates. Its gradient is the matrix

$$\nabla \Psi(x) = A(x_1, x_2) + x_3(\partial_1 a_3(x_1, x_2) | \partial_2 a_3(x_1, x_2) | 0). \quad (7)$$

Naturally, this gradient is everywhere nonsingular as soon as ε is small enough. In the context of nonlinear elasticity, the mapping Ψ^{-1} can also be viewed as a change of reference configuration for the shell (Ψ is orientation preserving).

3. The three-dimensional and rescaled problems

We assume that the shells are made of the same hyperelastic homogeneous material whose stored energy function is denoted by W . The function $W : M_3 \rightarrow \mathbb{R}$ is continuous and satisfies the growth and coercivity hypotheses

$$\begin{cases} \exists C > 0, \exists p \in]1, +\infty[, \forall F \in M_3, |W(F)| \leq C(1 + \|F\|^p), \\ \exists \alpha > 0, \exists \beta \geq 0, \forall F \in M_3, W(F) \geq \alpha \|F\|^p - \beta, \\ \forall F, F' \in M_3, |W(F) - W(F')| \leq C(1 + \|F\|^{p-1} + \|F'\|^{p-1}) \|F - F'\|. \end{cases} \quad (8)$$

Assumption (8)₃ was not needed in the case of planar membranes considered in Le Dret and Raoult [1995]. It is however quite natural. In particular, if W is quasiconvex (8)₁ implies (8)₃, cf. Marcellini [1985]. Assumption (8)₃ also holds true if W is continuously differentiable and its derivative grows as $\|F\|^{p-1}$ at infinity.

Let $S_\varepsilon^\pm = \omega \times \{\pm\varepsilon\}$ and define $\tilde{S}_\varepsilon^\pm = \Psi(S_\varepsilon^\pm)$ to be the top and bottom surfaces of the shell. For simplicity, we assume that the shells are solely submitted to the action of dead loading surface traction densities $\tilde{g}^\varepsilon \in L^q(\tilde{S}_\varepsilon^\pm; \mathbb{R}^3)$ with $1/p + 1/q = 1$. Taking body forces and lateral forces into account is straightforward. An example of live loads is detailed in the appendix. Let $\Gamma_\varepsilon = \partial\omega \times]-\varepsilon, \varepsilon[$ and $\tilde{\Gamma}_\varepsilon = \Psi(\Gamma_\varepsilon)$ be the lateral surface of $\tilde{\Omega}_\varepsilon$. We assume that the deformations of the shells satisfy a boundary condition of place on $\tilde{\Gamma}_\varepsilon$. The equilibrium problem may be formulated as a minimization problem:

$$\text{Find } \tilde{\phi}^\varepsilon \in \tilde{\Phi}_\varepsilon \text{ such that } \tilde{I}_\varepsilon(\tilde{\phi}^\varepsilon) = \inf_{\tilde{\varphi} \in \tilde{\Phi}_\varepsilon} \tilde{I}_\varepsilon(\tilde{\varphi}), \quad (9)$$

where the total energy \tilde{I}_ε is

$$\tilde{I}_\varepsilon(\tilde{\varphi}) = \int_{\tilde{\Omega}_\varepsilon} W(\nabla \tilde{\varphi}) dx - \int_{\tilde{S}_\varepsilon} \tilde{g}^\varepsilon \cdot \tilde{\varphi} d\tilde{\sigma}_\varepsilon, \quad (10)$$

$d\tilde{\sigma}_\varepsilon$ is the surface element on \tilde{S}_ε and the set of admissible deformations is

$$\tilde{\Phi}_\varepsilon = \{\tilde{\varphi} \in W^{1,p}(\tilde{\Omega}_\varepsilon; \mathbb{R}^3); \tilde{\varphi}(x) = x \text{ on } \tilde{\Gamma}_\varepsilon\}. \quad (11)$$

See Wang and Truesdell [1973], Marsden and Hughes [1983] or Ciarlet [1988], among others, for general references on three-dimensional nonlinear elasticity. A key-ingredient in existence proofs using the direct method of the calculus of variations is the sequential weak lower semi-continuity of the energy functional \tilde{I}_ε on $W^{1,p}(\tilde{\Omega}_\varepsilon; \mathbb{R}^3)$. Under assumptions (8), it is known that the energy functional \tilde{I}_ε in problem (9) is sequentially weakly lower semi-continuous on $W^{1,p}(\tilde{\Omega}_\varepsilon; \mathbb{R}^3)$ if and only if the function W is quasiconvex, *i.e.*,

$$\forall F \in M_3, \forall \theta \in W_0^{1,\infty}(O; \mathbb{R}^3), \int_O W(F + \nabla\theta(x)) dx \geq (\text{meas } O)W(F), \quad (12)$$

where O is any bounded domain of \mathbb{R}^3 , see Morrey [1952], Acerbi and Fusco [1984], Dacorogna [1989] and the references therein. Problem (9) was solved in the more physical case $W(F) = +\infty$ if $\det F \leq 0$ and $W(F) \rightarrow +\infty$ when $\det F \rightarrow 0^+$ by Ball [1977], under an assumption of polyconvexity of W , a notion more restrictive than quasiconvexity, plus appropriate growth and coercivity assumptions. For our purposes here, it is not desirable to assume at the onset that W is quasiconvex or polyconvex. There are two reasons for this. First of all, the zero thickness limit model we obtain always involves a quasiconvexification, which has to be effected whether W is quasiconvex or not. Secondly, we do not want to rule out important examples, such as the Saint Venant-Kirchhoff stored energy function which is neither polyconvex nor quasiconvex, see Raoult [1986]. Consequently, we *do not* assume that W is quasiconvex and problem (9) may well not possess any solutions. Naturally, if it does have solutions which are thus actual equilibrium deformations of the bodies, our results apply to these deformations.

Let us thus be given a diagonal minimizing sequence $\tilde{\phi}^\varepsilon$ for the sequence of energies \tilde{I}_ε over the sets $\tilde{\Phi}_\varepsilon$. More specifically, we assume that

$$\tilde{\phi}^\varepsilon \in \tilde{\Phi}_\varepsilon, \quad \tilde{I}_\varepsilon(\tilde{\phi}^\varepsilon) \leq \inf_{\tilde{\varphi} \in \tilde{\Phi}_\varepsilon} \tilde{I}_\varepsilon(\tilde{\varphi}) + \varepsilon h(\varepsilon), \quad (13)$$

where h is a positive function such that $h(\varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$. Such a sequence always exists and, if the minimization problems have solutions, $\tilde{\phi}^\varepsilon$ may be chosen to be such a solution.

As in the case of a planar membrane, we assume that $\|\tilde{g}^\varepsilon\|_{L^q(\tilde{S}_\varepsilon^\pm; \mathbb{R}^3)} \leq C\varepsilon$ where the constant C does not depend on ε . If we also considered body force densities or lateral traction densities, we would assume them to be of the order of 1 so that all force resultants would be of the order of ε (in particular, the weight of the material is allowed). This is essential in order to obtain a membrane model in the limit, see Le Dret and Raoult [1995], Fox, Raoult and Simo [1993] for a discussion of this observation.

We first rewrite problem (9) in the curvilinear coordinate system or, equivalently, we consider Ω_ε as a new reference configuration. Note that this configuration is not homogeneous anymore. If $\tilde{\varphi}$ is a deformation of the shell in its first reference configuration, we thus define for almost all $x \in \Omega_\varepsilon$,

$$\varphi(x) = \tilde{\varphi}(\Psi(x)), \quad (14)$$

and the set of admissible deformations becomes

$$\Phi_\varepsilon = \{\varphi \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3); \varphi(x) = \Psi(x) \text{ on } \Gamma_\varepsilon\}. \quad (15)$$

Similarly, we set for almost all $x \in S_\varepsilon^\pm$

$$g^\varepsilon(x) = \tilde{g}^\varepsilon(\Psi(x)), \quad (16)$$

and by defining $I_\varepsilon(\varphi) = \tilde{I}_\varepsilon(\tilde{\varphi})$ to be the energy in the new reference configuration, *i.e.*,

$$I_\varepsilon(\varphi) = \int_{\Omega_\varepsilon} W(\nabla\varphi(x)\nabla\Psi(x)^{-1}) \det(\nabla\Psi(x)) dx - \int_{S_\varepsilon^\pm} g^\varepsilon(x) \cdot \varphi(x) \|\text{cof } \nabla\Psi(x)e_3\| d\sigma_\varepsilon, \quad (17)$$

we obtain

$$\phi^\varepsilon \in \Phi_\varepsilon, \quad I_\varepsilon(\phi^\varepsilon) \leq \inf_{\varphi \in \Phi_\varepsilon} I_\varepsilon(\varphi) + \varepsilon h(\varepsilon). \quad (18)$$

All these definitions may also be rewritten in terms of displacements $v(x) = \varphi(x) - \Psi(x)$.

We now are in a position to rescale the problem. Let $\Omega = \Omega_1$, $\Gamma = \Gamma_1$ and $S^\pm = S_1^\pm$ and define a rescaling operator Θ_ε by $(\Theta_\varepsilon\varphi)(x_1, x_2, x_3) = \varphi(x_1, x_2, \varepsilon x_3)$. Let $\phi(\varepsilon) = \Theta_\varepsilon\phi^\varepsilon$, $\Psi(\varepsilon)(x) = \Theta_\varepsilon\Psi$ and $g(\varepsilon)(x) = \Theta_\varepsilon g^\varepsilon$. The rescaled displacement $u(\varepsilon) = \phi(\varepsilon) - \Psi(\varepsilon)$ belongs to $V = W_\Gamma^{1,p}(\Omega; \mathbb{R}^3)$. We rescale the energies by setting $I(\varepsilon)(\varphi) = \varepsilon^{-1}I_\varepsilon(\Theta_\varepsilon^{-1}\varphi)$, *i.e.*,

$$I(\varepsilon)(\varphi) = \int_\Omega W\left(\left(\partial_1\varphi \middle| \partial_2\varphi \middle| \frac{\partial_3\varphi}{\varepsilon}\right)A(\varepsilon)^{-1}\right) \det A(\varepsilon) dx - \int_{S^\pm} \varepsilon^{-1}g(\varepsilon) \cdot \varphi \|\text{cof } A(\varepsilon)e_3\| d\sigma, \quad (19)$$

with the notation:

$$A(\varepsilon)(x) = \nabla\Psi(x_1, x_2, \varepsilon x_3) = A(x_1, x_2) + \varepsilon x_3(\partial_1 a_3(x_1, x_2) | \partial_2 a_3(x_1, x_2) | 0). \quad (20)$$

In terms of the rescaled displacements, the rescaled energy reads:

$$\begin{aligned} J(\varepsilon)(v) &= \int_\Omega W\left(\left(\partial_1 v \middle| \partial_2 v \middle| \frac{\partial_3 v}{\varepsilon}\right)A(\varepsilon)^{-1} + I\right) \det A(\varepsilon) dx \\ &\quad - \int_{S^\pm} \varepsilon^{-1}g(\varepsilon) \cdot (\Psi(\varepsilon) + v) \|\text{cof } A(\varepsilon)e_3\| d\sigma. \end{aligned} \quad (21)$$

It is immediate that

$$J(\varepsilon)(u(\varepsilon)) \leq \inf_{v \in V} J(\varepsilon)(v) + h(\varepsilon). \quad (22)$$

For notational brevity, we also introduce the rescaled elastic energy:

$$E(\varepsilon)(v) = \int_{\Omega} W\left(\left(\partial_1 v \middle| \partial_2 v \middle| \frac{\partial_3 v}{\varepsilon}\right) A(\varepsilon)^{-1} + I\right) \det A(\varepsilon) dx, \quad (23)$$

and the rescaled virtual work of the applied loads:

$$L(\varepsilon)(v) = \int_{S^{\pm}} \varepsilon^{-1} g(\varepsilon) \cdot (\Psi(\varepsilon) + v) \|\operatorname{cof} A(\varepsilon) e_3\| d\sigma. \quad (24)$$

The assumed bound on \tilde{g}^{ε} ensures that $\|\varepsilon^{-1} g(\varepsilon)\|_{L^q(S^{\pm}; \mathbb{R}^3)} \leq C$. We may thus assume that there exists $g \in L^q(S^{\pm}; \mathbb{R}^3)$ such that

$$\varepsilon^{-1} g(\varepsilon) \rightharpoonup g \quad \text{in} \quad L^q(S^{\pm}; \mathbb{R}^3)$$

by extracting a subsequence, if necessary. Examples of such loadings are for instance dead loadings normal to the reference configurations of the shell, of the form $g^{\varepsilon}(x) = \varepsilon h^+(x_1, x_2) a_3(x_1, x_2)$ if $x_3 = \varepsilon$ and $g^{\varepsilon}(x) = \varepsilon h^-(x_1, x_2) a_3(x_1, x_2)$ if $x_3 = -\varepsilon$. See the Appendix for a concise treatment of pressure loads.

4. Computation of the Γ -limit of the rescaled energies

We use Γ -convergence theory to determine the asymptotic behavior of the rescaled displacements $u(\varepsilon)$ when $\varepsilon \rightarrow 0$. In the sequel, the thickness parameter ε will take its values in a sequence $\varepsilon_n \rightarrow 0$. Since the results do not depend on the sequence in question, and for notational brevity, we will simply use the notation ε . Let us recall that a sequence of functions G_{ε} from a metric space X into $\bar{\mathbb{R}}$ is said to Γ -converge toward G_0 for the topology of X if the following two conditions are satisfied for all $x \in X$:

$$\begin{cases} \forall x_{\varepsilon} \rightarrow x, \liminf G_{\varepsilon}(x_{\varepsilon}) \geq G_0(x), \\ \exists y_{\varepsilon} \rightarrow x, G_{\varepsilon}(y_{\varepsilon}) \rightarrow G_0(x). \end{cases}$$

If the sequence G_{ε} Γ -converges, its Γ -limit is alternatively given by

$$G_0(x) = \min\{\liminf G_{\varepsilon}(x_{\varepsilon}); x_{\varepsilon} \rightarrow x\}.$$

In addition, the set of functions from X into $\bar{\mathbb{R}}$ has a sequential compactness property with respect to Γ -convergence in the sense that any sequence $G_{\varepsilon} : X \rightarrow \bar{\mathbb{R}}$ admits a Γ -convergent subsequence. The main interest of Γ -convergence is that if the minimizers of G_{ε} stay in a compact set of X for all ε , then their limit points are minimizers of G_0 , see De Giorgi and Franzoni [1975], Attouch [1984], Dal Maso [1993].

We extend the energies to $L^p(\Omega; \mathbb{R}^3)$ by setting

$$\forall v \in L^p(\Omega; \mathbb{R}^3), \quad J^*(\varepsilon)(v) = \begin{cases} J(\varepsilon)(v) & \text{if } v \in V, \\ +\infty & \text{otherwise.} \end{cases} \quad (25)$$

Let us now proceed to compute the Γ -limit of the sequence $J^*(\varepsilon)$ for the strong topology of $L^p(\Omega; \mathbb{R}^3)$. Let $M_{3,2}$ be the space of 3×2 real matrices endowed with the usual Euclidean norm $\|\bar{F}\| = \sqrt{\text{tr}(\bar{F}^T \bar{F})}$. We note $(z_1|z_2)$ the matrix of $M_{3,2}$ whose α -th column is composed of the components of $z_\alpha \in \mathbb{R}^3$ in the canonical basis. For all $\bar{F} = (z_1|z_2) \in M_{3,2}$ and $z \in \mathbb{R}^3$, we also note $(\bar{F}|z)$ the matrix whose first two columns are z_1 and z_2 and whose third column is z .

Let us introduce a function $W_0: \bar{\omega} \times M_{3,2} \rightarrow \mathbb{R}$

$$W_0(x, \bar{F}) = \inf_{z \in \mathbb{R}^3} W((\bar{F}|z)A^{-1}(x)). \quad (26)$$

Due to the coercivity assumption (8)₂, it is clear that this function is well defined. Besides, since W is continuous, the infimum is attained. Let us briefly state a few properties of W_0 : The function W_0 is continuous on $\bar{\omega} \times M_{3,2}$ and satisfies the growth and coercivity estimates

$$\begin{cases} \exists C' > 0, \forall \bar{F} \in M_{3,2}, \forall x \in \bar{\omega}, |W_0(x, \bar{F})| \leq C'(1 + \|\bar{F}\|^p), \\ \exists \alpha' > 0, \exists \beta' \geq 0, \forall \bar{F} \in M_{3,2}, \forall x \in \bar{\omega}, W_0(x, \bar{F}) \geq \alpha' \|\bar{F}\|^p - \beta'. \end{cases} \quad (27)$$

See Le Dret and Raoult [1995] for a proof in the planar case. We use here in addition the continuity of A and A^{-1} on $\bar{\omega}$.

Let $QW_0 = \sup\{Z: \bar{\omega} \times M_{3,2} \rightarrow \mathbb{R}, Z \text{ quasiconvex}, Z \leq W_0\}$ be the quasiconvex envelope of W_0 , see Dacorogna [1982] for the definition and properties of quasiconvex functions and quasiconvex envelopes. Recall that a function Z of x and \bar{F} is quasiconvex if it satisfies

$$\forall x_0 \in \bar{\omega}, \forall \bar{F} \in M_{3,2}, \forall \theta \in W_0^{1,\infty}(O; \mathbb{R}^3), \int_O Z(x_0, \bar{F} + \nabla \theta(x)) dx \geq (\text{meas } O)Z(x_0, \bar{F}), \quad (28)$$

where O is any bounded domain of \mathbb{R}^2 . This is the same definition as (12) in the 3×2 case with the variable x_0 frozen. We introduce the space

$$V_M = \{v \in V; \partial_3 v = 0\}, \quad (29)$$

which we call the space of membrane displacements. This space is canonically isomorphic to $W_0^{1,p}(\omega; \mathbb{R}^3)$ and we let \bar{v} denote the element of $W_0^{1,p}(\omega; \mathbb{R}^3)$ that is associated with $v \in V_M$ through this isomorphism. The expression of the Γ -limit of the sequence $J^*(\varepsilon)$ is given in the following theorem.

THEOREM 1. — *The sequence $J^*(\varepsilon)$ Γ -converges for the strong topology of $L^p(\Omega; \mathbb{R}^3)$ when $\varepsilon \rightarrow 0$. Let $J^*(0)$ be its Γ -limit. For all $v \in L^p(\Omega; \mathbb{R}^3)$, $J^*(0)(v)$ is given by*

$$J^*(0)(v) = \begin{cases} 2 \int_\omega QW_0(x, (a_1 + \partial_1 \bar{v}|a_2 + \partial_2 \bar{v}))\sqrt{a} dx_1 dx_2 \\ \quad - \int_\omega \mathcal{G} \cdot (\psi + \bar{v})\sqrt{a} dx_1 dx_2 & \text{if } v \in V_M, \\ +\infty & \text{otherwise,} \end{cases} \quad (30)$$

where $\mathcal{G}(x_1, x_2) = g(x_1, x_2, 1) + g(x_1, x_2, -1)$.

For clarity, we break the proof of Theorem 1 into a series of lemmas. We will return to the mechanical interpretation of Theorem 1 in the next section. Let us first give a few simple convergence results for the various geometrical quantities associated with the shells.

LEMMA 2. — *The matrix $A(\varepsilon)$ satisfies*

$$\det A(\varepsilon) = \|\operatorname{cof} A(\varepsilon)e_3\| \rightarrow \sqrt{a} \text{ in } C^0(\bar{\Omega}), \quad A(\varepsilon)^{-1} \rightarrow A^{-1} \text{ in } C^0(\bar{\Omega}; M_3) \quad (31)$$

and the rescaled chart $\Psi(\varepsilon)$ satisfies

$$\Psi(\varepsilon) \rightarrow \Psi(0) \text{ in } C^0(\bar{\Omega}; \mathbb{R}^3) \quad (32)$$

where $\Psi(0)(x_1, x_2, x_3) = \psi(x_1, x_2)$.

We now extract a Γ -convergent subsequence, still denoted $J^*(\varepsilon)$, and call $J^*(0)$ its Γ -limit. The uniqueness of $J^*(0)$ will make the extraction of this subsequence superfluous *a posteriori*.

LEMMA 3. — *Let $v(\varepsilon) \in L^p(\Omega; \mathbb{R}^3)$ be a sequence such that $J^*(\varepsilon)(v(\varepsilon)) \leq C < +\infty$ where C does not depend on ε . Then $v(\varepsilon)$ is uniformly bounded in V , $\varepsilon^{-1}\partial_3 v(\varepsilon)$ is uniformly bounded in $L^p(\Omega; \mathbb{R}^3)$ and the limit points of $v(\varepsilon)$ for the weak topology of V belong to V_M .*

Proof. — Consider a sequence $v(\varepsilon) \in L^p(\Omega; \mathbb{R}^3)$ such that

$$J^*(\varepsilon)(v(\varepsilon)) \leq C < +\infty. \quad (33)$$

The definition (25) of $J^*(\varepsilon)$ implies first of all that $v(\varepsilon) \in V$ for all $\varepsilon > 0$.

The following inequality is an easy consequence of Lemma 2:

$$\|FA(\varepsilon)^{-1} + I\|^p \geq c_1 \|F\|^p - c_2, \quad (34)$$

where $c_1 > 0$ and c_2 do not depend either on ε or on x . Furthermore, it is clear that for all $\varepsilon \leq 1$, $\|(z_1|z_2|\varepsilon^{-1}z_3)\| \geq \|(z_1|z_2|z_3)\|$. It follows then from the coercivity of the function W , estimate (34) and Lemma 2 that there exists constants $c_3 > 0$ and c_4 such that

$$J^*(\varepsilon)(v(\varepsilon)) \geq c_3 \|\nabla v(\varepsilon)\|_{L^p(\Omega; M_3)}^p - c_4 - \|\varepsilon^{-1}g(\varepsilon)\|_{L^q(S^\pm; \mathbb{R}^3)} \|v(\varepsilon)\|_{W^{1,p}(\Omega; \mathbb{R}^3)}. \quad (35)$$

Therefore, Poincaré's inequality implies the desired uniform bound for $v(\varepsilon)$.

On the other hand, since $\|(z_1|z_2|\varepsilon^{-1}z_3)\| \geq \varepsilon^{-1}\|z_3\|$, it follows from inequalities (33) and (34) that $\|\partial_3 v(\varepsilon)\|_{L^p(\Omega; \mathbb{R}^3)} \leq c_5 \varepsilon$, so that $\partial_3 v(\varepsilon) \rightarrow 0$ strongly in $L^p(\Omega; \mathbb{R}^3)$. If we let v denote any limit point of the sequence $v(\varepsilon)$ for the weak topology of $W_\Gamma^{1,p}(\Omega; \mathbb{R}^3)$, it follows at once that $\partial_3 v = 0$, hence v belongs to V_M . \square

COROLLARY 4. — *If $v \in L^p(\Omega; \mathbb{R}^3)$ but $v \notin V_M$, then $J^*(0)(v) = +\infty$.*

Proof. — Indeed, if $J^*(0)(v) < +\infty$, there exists a sequence $v(\varepsilon)$ that converges strongly to v in $L^p(\Omega; \mathbb{R}^3)$ and such that $J^*(\varepsilon)(v(\varepsilon)) \rightarrow J^*(0)(v)$. By Lemma 3, $v(\varepsilon) \rightharpoonup v$ in V and $v \in V_M$. \square

We thus only have to compute the value of the Γ -limit for displacements in V_M . We first establish a bound from below for the Γ -limit functional.

PROPOSITION 5. — For all $v \in V_M$, we have that

$$J^*(0)(v) \geq 2 \int_{\omega} QW_0(x, (a_1 + \partial_1 \bar{v} | a_2 + \partial_2 \bar{v})) \sqrt{a} dx_1 dx_2 - \int_{\omega} \mathcal{G} \cdot (\psi + \bar{v}) \sqrt{a} dx_1 dx_2. \quad (36)$$

Proof. — Consider any $v \in V_M$. Since $J(\varepsilon)(v)$ is obviously bounded from above independently of ε , it follows that $J^*(0)(v) < +\infty$. By the definition of Γ -convergence, there exists a sequence $v(\varepsilon)$ such that $v(\varepsilon) \rightarrow v$ strongly in $L^p(\Omega; \mathbb{R}^3)$ and $J^*(\varepsilon)(v(\varepsilon)) \rightarrow J^*(0)(v)$, so that $v(\varepsilon) \in V$. Moreover, by Lemma 3, $v(\varepsilon) \rightharpoonup v$ weakly in V , hence its trace on S^\pm converges strongly in $L^p(S^\pm; \mathbb{R}^3)$. Since $\varepsilon^{-1}g(\varepsilon) \rightharpoonup g$ weakly in $L^q(S^\pm; \mathbb{R}^3)$, we thus have

$$L(\varepsilon)(v(\varepsilon)) \rightarrow L(0)(v) = \int_{S^\pm} g \cdot (\Psi(0) + v) \sqrt{a} d\sigma \quad (37).$$

Let us examine the asymptotic behavior of the rescaled elastic energy. Let $\varphi(\varepsilon) = \Psi(\varepsilon) + v(\varepsilon)$ be the deformation associated with displacement $v(\varepsilon)$. For any $F = (z_1 | z_2 | z_3) \in M_3$ and $x = (x_1, x_2, x_3) \in \bar{\Omega}$, we can write,

$$W((z_1 | z_2 | \varepsilon^{-1} z_3) A(\varepsilon)^{-1}(x)) = W((z_1 | z_2 | \varepsilon^{-1} z_3) A^{-1}(x_1, x_2)) + R(x, \varepsilon, F) \quad (38)$$

where, due to hypothesis (8)₃,

$$\begin{aligned} |R(x, \varepsilon, F)| \leq C \left(1 + \left\| \left(z_1 \mid z_2 \mid \frac{z_3}{\varepsilon} \right) A(\varepsilon)^{-1}(x) \right\|^{p-1} + \left\| \left(z_1 \mid z_2 \mid \frac{z_3}{\varepsilon} \right) A^{-1}(x_1, x_2) \right\|^{p-1} \right) \\ \times \left\| \left(z_1 \mid z_2 \mid \frac{z_3}{\varepsilon} \right) [A(\varepsilon)^{-1}(x) - A^{-1}(x_1, x_2)] \right\|. \end{aligned} \quad (39)$$

Since the matrix $A(\varepsilon)(x)$ is of the form $A(\varepsilon)(x_1, x_2, x_3) = A(x_1, x_2) + \varepsilon x_3 B(x_1, x_2)$, it follows that for ε small enough

$$A(\varepsilon)^{-1}(x_1, x_2, x_3) = A^{-1}(x_1, x_2)(I + \varepsilon S(\varepsilon, x)), \quad (40)$$

where $S(\varepsilon, \cdot)$ is bounded in $C^0(\bar{\Omega}; M_3)$ uniformly with respect to ε . Consequently,

$$|R(x, \varepsilon, F)| \leq C \varepsilon \left(1 + \left\| \left(z_1 \mid z_2 \mid \frac{z_3}{\varepsilon} \right) \right\|^p \right) \quad (41)$$

where C does not depend on x and on ε . If we replace F by $\nabla \varphi(\varepsilon)$ in (41) and integrate it on Ω against the weight $\det A(\varepsilon)$, we thus obtain by Lemmas 2 and 3

$$\int_{\Omega} |R(x, \varepsilon, \nabla \varphi(\varepsilon)(x))| \det A(\varepsilon) dx \leq C \varepsilon. \quad (42)$$

We now infer from equation (38) and the definition (26) of W_0 that

$$\begin{aligned}
E(\varepsilon)(v(\varepsilon)) &\geq \int_{\Omega} W_0((x_1, x_2), (\partial_1\varphi(\varepsilon)|\partial_2\varphi(\varepsilon))) \det A(\varepsilon) dx \\
&\quad + \int_{\Omega} R(x, \varepsilon, \nabla\varphi(\varepsilon)(x)) \det A(\varepsilon) dx \\
&\geq \int_{\Omega} QW_0((x_1, x_2), (\partial_1\varphi(\varepsilon)|\partial_2\varphi(\varepsilon))) \det A(\varepsilon) dx \\
&\quad + \int_{\Omega} R(x, \varepsilon, \nabla\varphi(\varepsilon)(x)) \det A(\varepsilon) dx.
\end{aligned} \tag{43}$$

Therefore, estimate (42) implies that

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} E(\varepsilon)(v(\varepsilon)) &\geq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} QW_0((x_1, x_2), (\partial_1\varphi(\varepsilon)|\partial_2\varphi(\varepsilon))) \det A(\varepsilon) dx \\
&= \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} QW_0((x_1, x_2), (\partial_1\varphi(\varepsilon)|\partial_2\varphi(\varepsilon))) \det A dx
\end{aligned} \tag{44}$$

since $\det A(\varepsilon) \rightarrow \det A$ in $C^0(\bar{\Omega})$. Let $G: W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow \mathbb{R}$ be defined by

$$G(\varphi) = \int_{\Omega} QW_0((x_1, x_2), (\partial_1\varphi|\partial_2\varphi)) \det A(x_1, x_2) dx. \tag{45}$$

We define a function $Z: \Omega \times M_3 \rightarrow \mathbb{R}$ by $Z(x, (z_1|z_2|z_3)) = QW_0((x_1, x_2), (z_1|z_2)) \det A(x_1, x_2)$ so that $G(\varphi) = \int_{\Omega} Z(x, \nabla\varphi(x)) dx$. Since QW_0 is quasiconvex, it is easy to see that Z is also quasiconvex, see Le Dret and Raoult [1995]. Moreover, Z is continuous, bounded below and satisfies the growth condition (8)₁ since QW_0 satisfies (27)₁. Therefore, the function G is sequentially weakly lower semi-continuous on $W^{1,p}(\Omega; \mathbb{R}^3)$, see Acerbi and Fusco [1984], Meyers [1965], Dacorogna [1989]. Consequently, as $\varphi(\varepsilon) \rightharpoonup \varphi = \Psi(0) + v$ in $W^{1,p}(\Omega; \mathbb{R}^3)$,

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} E(\varepsilon)(v(\varepsilon)) &\geq \liminf_{\varepsilon \rightarrow 0} G(\varphi(\varepsilon)) \geq G(\varphi) \\
&= 2 \int_{\omega} QW_0((x_1, x_2), (a_1 + \partial_1\bar{v}|a_2 + \partial_2\bar{v})) \sqrt{a(x_1, x_2)} dx_1 dx_2,
\end{aligned} \tag{46}$$

and the proof is complete. \square

Let us now turn to proving the reverse inequality. We first recall a technical lemma, see Dal Maso [1993], Le Dret and Raoult [1995].

LEMMA 6. — *Let $X \hookrightarrow Y$ be two Banach spaces such that X is reflexive and compactly embedded in Y . Consider a functional $G: X \rightarrow \mathbb{R}$ such that for all $v \in X$, $G(v) \geq g(\|v\|_X)$ where g is such that $g(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. Let $G^*: Y \rightarrow \mathbb{R}$ be defined by $G^*(v) = G(v)$ if $v \in X$, $G^*(v) = +\infty$ otherwise. Let Γ - G denote the sequential lower semi-continuous envelope*

of G for the weak topology of X and $\Gamma\text{-}G^*$ denote the lower semi-continuous envelope of G^* for the strong topology of Y . Then $\Gamma\text{-}G^* = (\Gamma\text{-}G)^*$.

PROPOSITION 7. — For all $v \in V_M$, the following estimate holds true:

$$J^*(0)(v) \leq 2 \int_{\omega} QW_0((x_1, x_2), (a_1 + \partial_1 \bar{v} | a_2 + \partial_2 \bar{v})) \sqrt{a} dx_1 dx_2 - \int_{\omega} \mathcal{G} \cdot (\psi + \bar{v}) \sqrt{a} dx_1 dx_2. \quad (47)$$

Proof. — Let us consider $v \in V_M$. For all $w \in W_0^{1,p}(\omega; \mathbb{R}^3)$, we define a displacement

$$v(\varepsilon)(x) = \bar{v}(x_1, x_2) + \varepsilon x_3 w(x_1, x_2), \quad (48)$$

and the associated deformation $\varphi(\varepsilon) = \Psi(\varepsilon) + v(\varepsilon) = \bar{\varphi} + \varepsilon x_3 (a_3 + w)$, with $\bar{\varphi} = \psi + \bar{v}$. Obviously, $v(\varepsilon) \rightarrow v$ strongly in $W^{1,p}(\Omega; \mathbb{R}^3)$. Let us examine the limit behavior of the sequence $J^*(\varepsilon)(v(\varepsilon))$. By the dominated convergence theorem and the growth estimate, it is clear that

$$\begin{aligned} E(\varepsilon)(v(\varepsilon)) &= \int_{\Omega} W((\partial_1 \varphi(\varepsilon) | \partial_2 \varphi(\varepsilon) | a_3 + w) A(\varepsilon)^{-1}) \det A(\varepsilon) dx \\ &\rightarrow 2 \int_{\omega} W((\partial_1 \bar{\varphi} | \partial_2 \bar{\varphi} | a_3 + w) A^{-1}) \det A dx_1 dx_2 \end{aligned} \quad (49)$$

when $\varepsilon \rightarrow 0$. Consequently,

$$J^*(\varepsilon)(v(\varepsilon)) \longrightarrow 2 \int_{\omega} W((\partial_1 \bar{\varphi} | \partial_2 \bar{\varphi} | a_3 + w) A^{-1}) \det A dx_1 dx_2 - \int_{\omega} \mathcal{G} \cdot (\psi + \bar{v}) \sqrt{a} dx_1 dx_2. \quad (50)$$

As this is true for all $w \in W_0^{1,p}(\omega; \mathbb{R}^3)$, it follows from the definition of Γ -convergence that

$$\begin{aligned} J^*(0)(v) \leq \inf_{w \in W_0^{1,p}(\omega; \mathbb{R}^3)} \left\{ 2 \int_{\omega} W((\partial_1 \bar{\varphi} | \partial_2 \bar{\varphi} | a_3 + w) A^{-1}) \det A dx_1 dx_2 \right\} \\ - \int_{\omega} \mathcal{G} \cdot (\psi + \bar{v}) \sqrt{a} dx_1 dx_2. \end{aligned} \quad (51)$$

We remark that in inequality (51), the infimum over $W_0^{1,p}(\omega; \mathbb{R}^3)$ can be replaced by the infimum over $L^p(\omega; \mathbb{R}^3)$, by the density of $W_0^{1,p}(\omega; \mathbb{R}^3)$ in $L^p(\omega; \mathbb{R}^3)$ and by the dominated convergence theorem. The function $g: \omega \times \mathbb{R}^3 \rightarrow \mathbb{R}$, $g(x, z) = W((\partial_1 \bar{\varphi} | \partial_2 \bar{\varphi} | a_3 + z) A^{-1})$ is a Carathéodory function. Hence, the measurable selection lemma, cf. Ekeland and Temam [1974], shows that there exists a measurable function w_0 such that

$$W_0(x, (\partial_1 \bar{\varphi}(x) | \partial_2 \bar{\varphi}(x))) = W((\partial_1 \bar{\varphi}(x) | \partial_2 \bar{\varphi}(x) | a_3(x) + w_0(x)) A^{-1}(x)) \quad (52)$$

for almost all $x \in \omega$. Due to the coercivity estimate, $w_0 \in L^p(\omega; \mathbb{R}^3)$ and thus

$$\inf_{w \in L^p(\omega; \mathbb{R}^3)} \left\{ \int_{\omega} W((\partial_1 \bar{\varphi} | \partial_2 \bar{\varphi} | a_3 + w) A^{-1}) \det A dx \right\} \leq \int_{\omega} W_0(x, (\partial_1 \bar{\varphi} | \partial_2 \bar{\varphi})) \det A dx. \quad (53)$$

Let $G: W_0^{1,p}(\omega; \mathbb{R}^3) \rightarrow \mathbb{R}$ be defined by

$$G(\bar{v}) = 2 \int_{\omega} W_0(x, (\partial_1 \bar{\varphi} | \partial_2 \bar{\varphi})) \sqrt{a} dx_1 dx_2 - \int_{\omega} \mathcal{G} \cdot (\psi + \bar{v}) \sqrt{a} dx_1 dx_2, \quad (54)$$

with $\bar{\varphi} = \psi + \bar{v}$. It follows from (53) that for all $v \in V_M$

$$J^*(0)(v) \leq G(\bar{v}). \quad (55)$$

Let G^* be defined on $L^p(\Omega; \mathbb{R}^3)$ by $G^*(v) = G(\bar{v})$ if $v \in V_M$, $G^*(v) = +\infty$ otherwise. Corollary 4 and (55) then imply that for all $v \in L^p(\Omega; \mathbb{R}^3)$

$$J^*(0)(v) \leq G^*(v). \quad (56)$$

Since $J^*(0)$ is lower semi-continuous on $L^p(\Omega; \mathbb{R}^3)$, it is smaller than the lower semi-continuous envelope of G^* . It is known, see Acerbi and Fusco [1984], that the sequential weak lower semi-continuous envelope Γ - G of G on $W_0^{1,p}(\omega; \mathbb{R}^3)$ is given by

$$\Gamma\text{-}G(\bar{v}) = 2 \int_{\omega} QW_0(x, (\partial_1 \bar{\varphi} | \partial_2 \bar{\varphi})) \sqrt{a} dx_1 dx_2 - \int_{\omega} \mathcal{G} \cdot (\psi + \bar{v}) \sqrt{a} dx_1 dx_2. \quad (57)$$

Therefore, Lemma 6 with $X = V_M$, $Y = L^p(\Omega; \mathbb{R}^3)$ and $g(t) = \alpha(t^p - 1)$ implies that

$$J^*(0) \leq \Gamma\text{-}G^* = (\Gamma\text{-}G)^*, \quad (58)$$

which proves the Proposition. \square

Proof of Theorem 1. — Use Corollary 4 for the case $v \notin V_M$ and Propositions 5 and 7 for the case $v \in V_M$. \square

5. The limit nonlinear membrane shell model

We now use Theorem 1 to characterize the asymptotic behavior of diagonal minimizing sequences of rescaled deformations $\phi(\varepsilon)$ satisfying $I(\varepsilon)(\phi(\varepsilon)) \leq \inf_{\varphi \in \Phi(\varepsilon)} I(\varepsilon)(\varphi) + h(\varepsilon)$ where h is a positive function such that $h(\varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$ and the sets of admissible deformations are

$$\Phi(\varepsilon) = \{\varphi \in W^{1,p}(\Omega; \mathbb{R}^3); \varphi(x) = \Psi(\varepsilon)(x) \text{ on } \Gamma\}.$$

We introduce the space of membrane shell deformations as $\Phi_M = \{\varphi \in W^{1,p}(\Omega; \mathbb{R}^3), \partial_3 \varphi = 0 \text{ in } \Omega, \varphi = \psi \text{ on } \Gamma\}$, which is isomorphic to the space $\bar{\Phi} = \{\bar{\varphi} \in W^{1,p}(\omega; \mathbb{R}^3), \bar{\varphi} = \psi \text{ on } \partial\omega\}$. We use the same notational device as for displacements to denote this isomorphism.

THEOREM 8. — *The sequence $\phi(\varepsilon)$ is relatively weakly compact in $W^{1,p}(\Omega; \mathbb{R}^3)$. Its limit points $\bar{\phi}$ belong to $\bar{\Phi}_M$ and are identified with elements $\bar{\phi}$ of $\bar{\Phi}$, solutions of the minimization problem $\bar{I}(0)(\bar{\phi}) = \inf_{\bar{\varphi} \in \bar{\Phi}} \bar{I}(0)(\bar{\varphi})$, where the membrane shell energy $\bar{I}(0)$ is given by*

$$\bar{I}(0)(\bar{\varphi}) = 2 \int_{\omega} QW_0(x, \nabla \bar{\varphi}(x)) \sqrt{a(x)} dx_1 dx_2 - \int_{\omega} \mathcal{G}(x) \cdot \bar{\varphi}(x) \sqrt{a(x)} dx_1 dx_2. \quad (59)$$

Moreover, $I(\varepsilon)(\phi(\varepsilon)) \rightarrow \bar{I}(0)(\bar{\phi})$ for all weakly convergent subsequences.

Proof. — See Le Dret and Raoult [1995], using the classical argument of De Giorgi. \square

Comments. — i) The limit energy depends on the deformation only through its first derivatives. In this sense, it is a membrane model with no bending or shear effects. Even if the three-dimensional material is homogenous in its reference configuration, the limit model exhibits a dependence on the point x . See Theorem 9 below for a more precise description of this dependence. Note that the limit minimization problem has a solution.

ii) If the function QW_0 is smooth enough, the Euler-Lagrange equations for the limit problem assume the form

$$-\frac{2}{\sqrt{a}} \partial_{\beta} \left[\left(\frac{\partial QW_0}{\partial F} (x, \nabla \bar{\phi}) \right)_{i\beta} \sqrt{a} \right] = \mathcal{G}_i \text{ in } \omega, \quad \bar{\phi}(x_1, x_2) = \psi(x_1, x_2) \text{ on } \partial\omega. \quad (60)$$

System (60) is a system of three second order quasilinear partial differential equations in the three unknowns $\bar{\phi}_i$.

See Le Dret and Raoult [1995] for more comments.

Theorem 8 gives information on the asymptotic behavior of the actual deformations $\tilde{\phi}^\varepsilon$ of the shell in its given reference configuration $\tilde{\Omega}_\varepsilon$, by reading it through the chart Ψ . However, we could have worked as well with a chart Ψ' associated with any other chart ψ' for \tilde{S} . More specifically, let O', e'_1, e'_2, e'_3 be another Cartesian frame in \mathbb{R}^3 , ω' a bounded open subset of the plane (O', e'_1, e'_2) and $\psi' : \omega' \rightarrow \tilde{S}$ a chart for \tilde{S} that satisfies the same hypotheses as ψ . With the deformation $\tilde{\phi}^\varepsilon$, we associate a new rescaled deformation $\phi'(\varepsilon)(x') = \tilde{\phi}^\varepsilon(\psi'(x'_1, x'_2) + \varepsilon x'_3 a'_3(x'_1, x'_2))$. Since $\lambda = \psi^{-1} \circ \psi'$ is a C^2 -diffeomorphism between ω' and ω , it is fairly clear that if $\bar{\phi} \in \bar{\Phi}_M$ is associated with a limit point of $\phi(\varepsilon)$, then $\bar{\phi}' = \bar{\phi} \circ \lambda$ is associated with a limit point of $\phi'(\varepsilon)$. Applying Theorem 8 in both charts, we see that $\bar{I}(0)(\bar{\phi}) = \bar{I}'(0)(\bar{\phi}')$ with obvious notation. This observation is confirmed by a direct computation. Indeed, let $W'_0: \bar{\omega}' \times M_{3,2} \rightarrow \mathbb{R}$ be defined by

$$W'_0((x'_1, x'_2), \bar{F}) = \inf_{z \in \mathbb{R}^3} W((\bar{F}|z)A'^{-1}(x'_1, x'_2))$$

where $A'(x'_1, x'_2) = (\partial_1 \psi' | \partial_2 \psi' | a'_3)(x'_1, x'_2)$. It is a simple matter to check that since the normal vectors are chart-independent, *i.e.*, $a'_3(x'_1, x'_2) = \pm a_3(x_1, x_2)$ whenever $(x_1, x_2) = \lambda(x'_1, x'_2)$,

then $W_0((x_1, x_2), \nabla \bar{\varphi}(x_1, x_2)) = W_0'((x'_1, x'_2), \nabla \bar{\varphi}'(x'_1, x'_2))$. Hence, the same holds true for the quasiconvex envelopes and thus for the energies themselves.

The above remarks demonstrate the intrinsic character of the limit minimization problem. It is nonetheless of prime importance to give an expression of the limit nonlinear shell problem in the original reference configuration of the shell, in particular if this configuration has special properties, for example is a natural configuration, a homogeneous configuration or an isotropic configuration. This is the object of the remainder of this section.

With any deformation $\bar{\varphi} \in \bar{\Phi}$ we thus associate a deformation of the shell in its reference configuration $\tilde{\varphi} = \bar{\varphi} \circ \psi^{-1} \in \tilde{\Phi}$ where

$$\tilde{\Phi} = \{\tilde{\varphi} \in W^{1,p}(\tilde{S}; \mathbb{R}^3); \tilde{\varphi}(\tilde{x}) = \tilde{x} \text{ on } \partial\tilde{S}\}.$$

Let $\tilde{\Pi}$ be the orthogonal projection on \tilde{S} , which is well defined in a tubular neighborhood of \tilde{S} . We extend the deformation to this tubular neighborhood by setting $\varphi(\tilde{x}) = \tilde{\varphi}(\tilde{\Pi}(\tilde{x}))$ and for $\tilde{x} \in \tilde{S}$, we let $D\tilde{\varphi}(\tilde{x}) = \nabla\varphi(\tilde{x})$. Therefore, $D\tilde{\varphi}(\tilde{x})$ is the 3×3 matrix of the components of $\nabla\varphi(\tilde{x})$ in the canonical Cartesian basis (e_1, e_2, e_3) . We will call this matrix the deformation gradient. We denote by $d\tilde{\sigma}$ the area element on \tilde{S} .

For all unit vectors $e \in S^2$, we choose a bounded open set $O_e \subset e^\perp$ and denote by Π_e the orthogonal projection on e^\perp . For all $\chi \in W_0^{1,\infty}(O_e; \mathbb{R}^3)$, we let $\chi_e(y) = \chi(\Pi_e(y))$ and for all $y \in O_e$, we define $D_{e^\perp}\chi(y) = \nabla\chi_e(y)$ which is again a 3×3 matrix. Then we have:

THEOREM 9. — *Let $\tilde{\phi}$ be a shell deformation associated with a minimizer $\bar{\phi}$ of the limit energy in the chart ψ , as in Theorem 8. Then $\tilde{\phi}$ is a solution of the minimization problem*

$$\tilde{I}_{\tilde{S}}(\tilde{\phi}) = \inf_{\tilde{\varphi} \in \tilde{\Phi}} \tilde{I}_{\tilde{S}}(\tilde{\varphi}). \quad (61)$$

The membrane shell energy $\tilde{I}_{\tilde{S}}$ is given by

$$\tilde{I}_{\tilde{S}}(\tilde{\varphi}) = 2 \int_{\tilde{S}} W_m(a_3(\tilde{x}), D\tilde{\varphi}(\tilde{x})) d\tilde{\sigma} - \int_{\tilde{S}} \tilde{\mathcal{G}} \cdot \tilde{\varphi} d\tilde{\sigma}, \quad (62)$$

where the elastic membrane stored energy function of the material $W_m: S^2 \times M_3 \rightarrow \mathbb{R}$ is defined by

$$W_m(e, F) = \inf_{\chi \in W_0^{1,\infty}(O_e; \mathbb{R}^3)} \left[\frac{1}{\text{meas } O_e} \int_{O_e} \left[\inf_{z \in \mathbb{R}^3} W(F + z \otimes e + D_{e^\perp}\chi(y)) \right] dy \right] \quad (63)$$

and $\tilde{\mathcal{G}}(\tilde{x}) = \mathcal{G}(\psi^{-1}(\tilde{x}))$.

Proof. — Recall that by Theorem 8, $\bar{\phi}$ minimizes the energy

$$\bar{I}(0)(\bar{\varphi}) = 2 \int_{\omega} QW_0(x, \nabla\bar{\varphi})\sqrt{a} dx_1 dx_2 - \int_{\omega} \mathcal{G} \cdot \bar{\varphi}\sqrt{a} dx_1 dx_2.$$

We start from Dacorogna's representation formula for the quasiconvex envelope of W_0 , cf. Dacorogna [1982, 1989], which states that

$$QW_0(x_0, \bar{F}) = \inf_{\bar{\chi} \in W_0^{1,\infty}(O; \mathbb{R}^3)} \left\{ \frac{1}{\text{meas } O} \int_O W_0(x_0, \bar{F} + \nabla \bar{\chi}(\bar{y})) d\bar{y} \right\}, \quad (64)$$

where O is a bounded open subset of \mathbb{R}^2 (this infimum does not depend on the choice of O). We choose $O = (D\psi(x_0))^{-1}(O_{a_3(\psi(x_0))})$. Due to the definition of W_0 , we thus have

$$QW_0(x_0, \bar{F}) = \inf_{\bar{\chi} \in W_0^{1,\infty}(O; \mathbb{R}^3)} \left\{ \frac{1}{\text{meas } O} \int_O \inf_{z \in \mathbb{R}^3} W((\bar{F} + \nabla \bar{\chi}(\bar{y})|z)A^{-1}(x_0)) d\bar{y} \right\}. \quad (65)$$

We now remark that

$$(\bar{F} + \nabla \bar{\chi}(\bar{y})|z)A^{-1}(x_0) = (\bar{F}|0)A^{-1}(x_0) + (0|z)A^{-1}(x_0) + (\nabla \bar{\chi}(\bar{y})|0)A^{-1}(x_0), \quad (66)$$

and we consider each of these three terms separately. First of all, if \bar{F} is a gradient, i.e., $\bar{F} = \nabla \bar{\varphi}(x_0)$, then $(\bar{F}|0)A^{-1}(x_0) = D\tilde{\varphi}(\psi(x_0))$. Indeed, if we let $\varphi(x_1, x_2, x_3) = \bar{\varphi}(x_1, x_2)$, then $\varphi(x) = \tilde{\varphi}(\tilde{\Pi}(\Psi(x)))$. Consequently, $(\nabla \bar{\varphi}(x_0)|0) = \nabla(\tilde{\varphi}(\tilde{\Pi}(\Psi(x_0))))\nabla\Psi(x_0) = D\tilde{\varphi}(\psi(x_0))A(x_0)$. ■

Secondly, letting $y = D\psi(x_0)\bar{y}$ and $\chi(y) = \bar{\chi}(\bar{y})$, we likewise note that $(\nabla \bar{\chi}(\bar{y})|0)A^{-1}(x_0) = D_{a_3(\psi(x_0))^\perp} \chi(y)$. Moreover, χ belongs to $W_0^{1,\infty}(O_{a_3(\psi(x_0))}; \mathbb{R}^3)$.

Finally, it is easily checked that $(0|z)A^{-1}(x_0) = z \otimes a_3(\psi(x_0))$. Replacing these expressions into (65), we obtain

$$\forall x_0 \in \omega, \quad QW_0(x_0, \nabla \bar{\varphi}(x_0)) = W_m(a_3(\psi(x_0)), D\tilde{\varphi}(\psi(x_0))) \quad (67)$$

from which Theorem 9 follows at once by the change of variables $x \mapsto \tilde{x} = \psi(x)$. □

Remarks. — i) The elastic membrane stored energy function depends on two variables: a unit vector and a matrix. The membrane shell energy is obtained by replacing the unit vector by the normal vector and the matrix by the deformation gradient. Note that deformation gradients always satisfy $D\tilde{\varphi}(\tilde{x})a_3(\tilde{x}) = 0$. Thus, expression (63) is only useful for couples (e, F) such that $Fe = 0$. In the planar case, the normal vector is constant and we recover the result of Le Dret and Raoult [1995]. The fact that the energy depends on the surface only through its normal vector is not directly apparent in expression (59) in terms of QW_0 .

ii) The limit energy (62) corresponds to the membrane part of the energy for inextensible one-director Cosserat shells obtained by Simo and Fox [1989]. However, let us point out that we do not make any *a priori* kinematic assumptions. Moreover, our analysis provides a convergence result and at the same time an exact formula for the constitutive law of the shell. Our model is a pure membrane model since it does not include shear and flexural effects.

iii) Note that $W_m(e, F) = W_m(-e, F)$ which is due to the fact that the energy does not depend on the orientation of the midsurface.

iv) Definition (63) does not depend on the choice of O_e in e^\perp .

v) Theorem 8 may be reformulated in terms of convergence of the deformations rescaled in the reference configuration, *i.e.*, defining $\tilde{\phi}(\varepsilon)(\tilde{x}) = \tilde{\phi}^\varepsilon(\tilde{\Pi}(\tilde{x}) + \varepsilon[(\tilde{x} - \tilde{\Pi}(\tilde{x})) \cdot a_3(\tilde{\Pi}(\tilde{x}))]a_3(\tilde{\Pi}(\tilde{x})))$ on $\tilde{\Omega}_1$ (assuming this is well defined, otherwise we just rescale on a thinner domain) then $\tilde{\phi}(\varepsilon) \rightharpoonup \tilde{\phi}$ in $W^{1,p}(\tilde{\Omega}_1; \mathbb{R}^3)$ where $\tilde{\phi}$ is a solution of problem (61)–(62).

6. Properties of the nonlinear membrane shell energy

In the Γ -convergence analysis, we have ignored the fact that the stored energy function W of the three-dimensional bodies has to satisfy material frame-indifference, since this was irrelevant for the convergence proof. In this section, we will investigate what are the consequences of material frame-indifference for the nonlinear membrane shell energy W_m as well as the consequences of material symmetry assumptions.

First of all, recall that the principle of material frame-indifference states that to be legitimate from the standpoint of continuum mechanics, a stored energy function W has to satisfy

$$\forall F \in M_3, \forall R \in SO(3), W(RF) = W(F), \quad (68)$$

see *e.g.* Ciarlet [1988], Wang and Truesdell [1973] or Marsden and Hughes [1983].

THEOREM 10. — *Let the stored energy function W satisfy the principle of material frame-indifference (68). Then, the nonlinear membrane shell energy W_m is frame-indifferent as well, in the sense that*

$$\forall e \in S^2, \forall F \in M_3, \forall R \in SO(3), W_m(e, RF) = W_m(e, F), \quad (69)$$

and there exists a function $\mathscr{W}_m: S^2 \times \mathbb{S}_3^> \rightarrow \mathbb{R}$, where $\mathbb{S}_3^>$ is the set of 3×3 positive semi-definite symmetric matrices, such that

$$\forall e \in S^2, \forall F \in M_3, W_m(e, F) = \mathscr{W}_m(e, F^T F). \quad (70)$$

Proof. — Let $F \in M_3$ and $R \in SO(3)$ be arbitrary matrices. Since for all $e \in S^2$, $y \in O_e$, $\chi \in W_0^{1,\infty}(O_e; \mathbb{R}^3)$ and $z \in \mathbb{R}^3$,

$$\begin{aligned} W(RF + z \otimes e + D_{e^\perp} \chi(y)) &= W(R(F + R^T z \otimes e + R^T D_{e^\perp} \chi(y))) \\ &= W(F + (R^T z) \otimes e + D_{e^\perp}(R^T \chi)(y)), \end{aligned} \quad (71)$$

definition (63) shows that (69) holds true. The existence of the representation function \mathcal{W}_m in formula (70) is then classical. \square

Remarks. — i) The representation formula (70) is given for an arbitrary couple (e, F) in $S^2 \times M_3$. Since deformation gradients $D\tilde{\varphi}(\tilde{x}) = F(\tilde{x})$ always satisfy $F(\tilde{x})a_3(\tilde{x}) = 0$, the associated strain tensors $C(\tilde{x}) = F(\tilde{x})^T F(\tilde{x})$ also satisfy $C(\tilde{x})a_3(\tilde{x}) = 0$. Thus, formula (70) is only useful for couples (e, C) such that $Ce = 0$.

ii) If F is the gradient of a smooth enough shell deformation $\tilde{\phi}$, the matrix $F(\tilde{x})^T F(\tilde{x})$ represents the metric of the deformed surface at point $\tilde{\phi}(\tilde{x})$. The membrane energy thus only depends on this metric, which is consistent with the intuition that the stress state in an elastic membrane depends only on the stretching that the deformed surface undergoes.

We now show that, due to frame indifference, if the three-dimensional stored energy function has a global minimum at $F = I$, the corresponding nonlinear shell energy is constant under compression. This means that it is possible to crumple a membrane shell without using any energy. This phenomenon was first noticed in the case of nonlinear strings by Acerbi, Buttazzo and Percivale [1991], then in the case of planar membranes by Percivale [1991] for isotropic materials and Le Dret and Raoult [1993, 1995] for general materials. The proof given in the latter article does not extend to the case of shells. The proof we provide below is at the same time simpler and more general. We note $v_i(F)$, $i = 1, 2, 3$, the singular values of F numbered in increasing order.

COROLLARY 11. — *Assume that the three-dimensional stored energy function W is such that $W(I) = 0$ and $W(F) \geq 0$ for all $F \in M_3$. Then, $W_m(e, F) = 0$ for all $F \in M_3$ such that $Fe = 0$ and $v_3(F) \leq 1$.*

Proof. — Fix $e \in S^2$. Since $W \geq 0$, it follows immediately that $W_m(e, F) \geq 0$ for all $F \in M_3$.

Let $F \in M_3$ be such that $Fe = 0$ and $v_3(F) \leq 1$. Let $U = \sqrt{F^T F}$. We can choose an orthonormal basis e, f_2, f_3 of eigenvectors of U , where e is associated with the eigenvalue 0 and f_i , $i = 2, 3$ are associated with the eigenvalues $v_i(F)$, $i = 2, 3$. It follows from (70) and the polar factorization theorem that $W_m(e, F) = W_m(e, U)$. It thus suffices to prove that $W_m(e, U) = 0$.

We consider the 1-periodic functions $\theta_i: \mathbb{R} \rightarrow \mathbb{R}$, $i = 2, 3$, defined by their restriction to $[0, 1[$

$$\theta_i(t) = \begin{cases} (1 - v_i(F))t & \text{if } 0 \leq t \leq \frac{1+v_i(F)}{2}, \\ (-1 - v_i(F))(t - 1) & \text{if } \frac{1+v_i(F)}{2} \leq t < 1. \end{cases}$$

Note that since $v_i(\bar{F}) \in [0, 1]$, $\frac{1+v_i(\bar{F})}{2} \in [0, 1]$ and the functions θ_i are well defined and belong to $W^{1,\infty}(\mathbb{R})$. We let for $y \in \mathbb{R}^3$

$$\chi_n(y) = \sum_{i=2}^3 \frac{1}{n} \theta_i(ny \cdot f_i) f_i. \tag{72}$$

Therefore,

$$\begin{aligned}\nabla\chi_n(y) &= \sum_{i=2}^3 \theta'_i(ny \cdot f_i) f_i \otimes f_i \\ &= \sum_{i=2}^3 (h_i^n(y) - v_i(F)) f_i \otimes f_i\end{aligned}\tag{73}$$

where h_i^n only takes the values ± 1 .

Without loss of generality, we may assume that $\text{meas } O_e = 1$. We introduce a smooth cut-off function $0 \leq \rho_n \leq 1$ defined on O_e and such that $\rho_n(y) = 1$ if $d(y, \partial O_e) \geq 1/n$, $\rho_n(y) = 0$ if $y \in \partial O_e$ and that $\|\nabla \rho_n\| \leq 2n$. Since $\rho_n \chi_n \in W_0^{1,\infty}(O_e; \mathbb{R}^3)$, we can use it in definition (63). It follows from this definition that for all measurable functions $h: O_e \rightarrow \{-1, 1\}$,

$$W_m(e, U) \leq \int_{O_e} W(U + h(y)e \otimes e + D_{e^\perp}(\rho_n \chi_n)(y)) dy.\tag{74}$$

Since $(\rho_n \chi_n)_e(y) = \chi_n(\Pi_e(y))$ for all $y \in \mathbb{R}^3$ such that $d(\Pi_e(y), \partial O_e) \geq 1/n$, we see that $D_{e^\perp}((\rho_n \chi_n))(y) = \nabla \chi_n(y)$ for all $y \in O_e$ such that $d(y, \partial O_e) \geq 1/n$. Therefore, since $U = \sum_{i=2}^3 v_i(F) f_i \otimes f_i$, we obtain

$$\begin{aligned}W_m(e, U) &\leq \int_{d(y, \partial O_e) \geq 1/n} W(h(y)e \otimes e + h_2^n(y) f_2 \otimes f_2 + h_3^n(y) f_3 \otimes f_3) dy \\ &\quad + \int_{d(y, \partial O_e) < 1/n} W(U + h(y)e \otimes e + D_{e^\perp}(\rho_n \chi_n)(y)) dy.\end{aligned}\tag{75}$$

Fix n . We choose $h(y)$ so that $(h(y)e \otimes e + h_2^n(y) f_2 \otimes f_2 + h_3^n(y) f_3 \otimes f_3) \in SO(3)$ for almost all y , which is obviously possible. With this choice, the first integral vanishes by frame indifference. It is clear that the integrand of the second term is bounded independently of n . Since $\text{meas } \{d(y, \partial O_e) < 1/n\} \rightarrow 0$ as $n \rightarrow +\infty$, we obtain $W(e, U) \leq 0$. \square

We now investigate the consequences of isotropy on the membrane energy. Recall first that an elastic material is said to be isotropic if

$$\forall F \in M_3, \forall R \in SO(3), W(FR) = W(F).\tag{76}$$

We show below that isotropy added to the principle of material indifference implies that the shell energy W_m does not depend on the normal vector.

THEOREM 12. — *Assume that the stored energy function W is isotropic (76). Then, the nonlinear membrane shell energy W_m is isotropic as well, in the sense that*

$$\forall e \in S^2, \forall F \in M_3, \forall R \in SO(3), W_m(e, F) = W_m(R^T e, FR).\tag{77}$$

If W furthermore satisfies the principle of material indifference, there exists a symmetric function $w_m: (\mathbb{R}_+)^2 \rightarrow \mathbb{R}$ such that

$$\forall e \in S^2, \forall F \in M_3, Fe = 0, \quad W_m(e, F) = w_m(v_2(F), v_3(F)). \quad (78)$$

Proof. — We may assume without loss of generality that for all $e \in S^2$ and all $R \in SO(3)$, $RO_e = O_{Re}$. If $\chi \in W_0^{1,\infty}(O_e; \mathbb{R}^3)$, the function χ_R defined by $\chi_R(y) = \chi(Ry)$ belongs to $W_0^{1,\infty}(O_{R^T e}; \mathbb{R}^3)$. Since for all $F \in M_3$, $e \in S^2$, $y \in O_e$, $\chi \in W_0^{1,\infty}(O_e; \mathbb{R}^3)$ and $z \in \mathbb{R}^3$,

$$\begin{aligned} W(FR + z \otimes (R^T e) + D_{(R^T e)^\perp} \chi_R(y)) &= W((F + z \otimes e + D_{e^\perp} \chi(Ry))R) \\ &= W(F + z \otimes e + D_{e^\perp} \chi(Ry)), \end{aligned} \quad (79)$$

definition (63) shows that (77) holds true.

Assume now that W satisfies the principle of material indifference. By theorem 10, $W_m(e, F) = \mathscr{W}_m(e, F^T F) = \mathscr{W}_m(R^T e, R^T F^T F R)$. In particular, for all $C \in \mathbb{S}_3^\geq$ and all $R \in SO(3)$ such that $R^T e = e$,

$$\mathscr{W}_m(e, C) = \mathscr{W}_m(e, R^T C R). \quad (80)$$

Consider now two matrices C and C' of \mathbb{S}_3 such that $Ce = C'e = 0$ and have the same eigenvalues. Proceeding as in Gurtin [1981], we see that there exists $R \in SO(3)$ with $R^T e = e$ such that $R^T C R = C'$. Consequently, there exists a function $w: S^2 \times (\mathbb{R}_+)^2 \rightarrow \mathbb{R}$, symmetric with respect to the last two arguments, such that for all C with $Ce = 0$

$$\mathscr{W}_m(e, C) = w(e, \lambda_2(C)^{1/2}, \lambda_3(C)^{1/2}), \quad (81)$$

where $\lambda_2(C)$, $\lambda_3(C)$ are the largest eigenvalues of C .

Let us now prove that w does not depend on e . Consider thus two unit vectors e and e' and let $R \in SO(3)$ be such that $e' = R^T e$. For all F such that $Fe = 0$, we have

$$W_m(e, F) = w(e, v_2(F), v_3(F)) \quad (82)$$

by (70) and (81). On the other hand, by (77),

$$W_m(e, F) = W_m(e', FR). \quad (83)$$

Since $FR e' = 0$ we also have

$$W_m(e', FR) = w(e', v_2(FR), v_3(FR)) \quad (84)$$

by (70) and (81) again. Since $v_i(FR) = v_i(F)$ for $i = 2, 3$, we conclude that for all $(v_2, v_3) \in (\mathbb{R}_+)^2$,

$$w(e, v_2, v_3) = w(e', v_2, v_3) = w_m(v_2, v_3), \quad (85)$$

which defines w_m and completes the proof. \square

Remarks. — i) In this case, the shell energy (62) assumes the form

$$\tilde{I}_{\tilde{S}}(\tilde{\varphi}) = 2 \int_{\tilde{S}} w_m(v_2(D\tilde{\varphi}(\tilde{x})), v_3(D\tilde{\varphi}(\tilde{x}))) d\tilde{\sigma} - \int_{\tilde{S}} \tilde{\mathcal{G}} \cdot \tilde{\varphi} d\tilde{\sigma}. \quad (86)$$

This formula applies to any surface, in particular to planar ones as in Le Dret and Raoult [1995]. Consequently, if the planar membrane energy is explicitly known, the shell energy is also explicitly determined without further computations. This is the case for the Saint Venant-Kirchhoff material

$$W(F) = \frac{\mu}{4} \text{tr}(F^T F - I)^2 + \frac{\lambda}{8} (\text{tr}(F^T F - I))^2$$

where $\mu > 0$ and $\lambda \geq 0$ are the Lamé moduli. We thus obtain according to Le Dret and Raoult [1995]:

$$\begin{aligned} W_m(F) &= \frac{E}{8} [v_3(F)^2 - 1]_+^2 + \frac{E}{8(1-\nu^2)} [v_2(F)^2 + \nu v_3(F)^2 - (1+\nu)]_+^2 \\ &+ \frac{E}{8(1-\nu^2)(1-2\nu)} [\nu(v_2(F)^2 + v_3(F)^2) - (1+\nu)]_+^2, \end{aligned} \quad (87)$$

where $[t]_+^2$ stands for $([t]_+)^2$ and E and ν are the Young modulus and the Poisson coefficient.

Appendix. A live loading case: the pressure load

Let us briefly show how the case of a quite realistic live loading can be handled. We assume that the upper and lower surfaces $\tilde{S}_\varepsilon^\pm$ of the shell are submitted to uniform hydrostatic pressures π_ε^\pm instead of prescribed dead loads. This means that the Cauchy stress vector on the deformed upper surface satisfies $Tn^+ = -\pi_\varepsilon^+ n^+$, where T is the Cauchy stress tensor, n^+ is the outer unit normal vector to the deformed upper surface and $\pi_\varepsilon^+ \in \mathbb{R}$, and similarly on the lower deformed surface. To be consistent with the order of magnitude of the loads we chose previously, we assume that $\pi_\varepsilon^\pm = \varepsilon \pi^\pm$, where π^\pm do not depend on ε . Let $\Delta\pi = \pi^+ - \pi^-$.

It is shown in Ball [1977], see also Sewell [1967], that the corresponding equilibrium problem may be formulated as an energy minimization problem as follows. Let $\tilde{\pi}_\varepsilon \in C^1(\overline{\tilde{\Omega}_\varepsilon})$ be such that $\tilde{\pi}_\varepsilon(\tilde{x}) = \pi_\varepsilon^+$ for $\tilde{x} \in \tilde{S}_\varepsilon^+$ and $\tilde{\pi}_\varepsilon(\tilde{x}) = \pi_\varepsilon^-$ for $\tilde{x} \in \tilde{S}_\varepsilon^-$. Then the pressure load equilibrium problem is, at least formally, equivalent to minimizing the energy

$$\tilde{I}_\varepsilon(\tilde{\varphi}) = \int_{\tilde{\Omega}_\varepsilon} W(\nabla \tilde{\varphi}) dx + P_\varepsilon(\tilde{\varphi}), \quad (88)$$

over the set of admissible deformations $\tilde{\Phi}_\varepsilon$, where

$$P_\varepsilon(\tilde{\varphi}) = \int_{\tilde{\Omega}_\varepsilon} \left[\tilde{\pi}_\varepsilon(\tilde{x}) \det \nabla \tilde{\varphi}(\tilde{x}) + \frac{1}{3} \nabla \tilde{\pi}_\varepsilon(\tilde{x}) \cdot (\text{adj } \nabla \tilde{\varphi}(\tilde{x}) \tilde{\varphi}(\tilde{x})) \right] d\tilde{x}, \quad (89)$$

and $\text{adj } F$ is the transpose of the cofactor matrix of F . We are at liberty to choose here $\tilde{\pi}_\varepsilon(\tilde{x}) = \frac{1}{2\varepsilon}[(\varepsilon + x_3(\tilde{x}))\pi_\varepsilon^+ + (\varepsilon - x_3(\tilde{x}))\pi_\varepsilon^-]$, with obvious notation.

For simplicity, we assume that the exponent p is strictly larger than 3, which trivially ensures that the energy (88) is bounded from below and has the same coercivity properties as before. This assumption also implies that there is no distinction between the distributional and the algebraic determinants and adjugates of the deformation gradients, which we thus all denote with a lowercase initial (see Ball [1977], Müller [1990], for a discussion of this question). Performing the same change of variables and rescaling as in section 3, we are thus led to the computation of the Γ -limit of the sequence of functionals:

$$J(\varepsilon)(v) = E(\varepsilon)(v) + P(\varepsilon)(v), \quad (90)$$

where

$$P(\varepsilon)(v) = \int_{\Omega} \left[\varepsilon \pi \det(\partial_1 \varphi | \partial_2 \varphi | \frac{\partial_3 \varphi}{\varepsilon}) + \frac{\Delta \pi}{6} e_3 \cdot (\text{adj}(\partial_1 \varphi | \partial_2 \varphi | \frac{\partial_3 \varphi}{\varepsilon}) \varphi) \right] dx, \quad (91)$$

with $\pi(x) = \frac{1}{2}[(1 + x_3)\pi^+ + (1 - x_3)\pi^-]$ and $\varphi = \Psi(\varepsilon) + v$ as usual. A simple algebraic calculation shows that

$$e_3 \cdot (\text{adj}(\partial_1 \varphi | \partial_2 \varphi | \frac{\partial_3 \varphi}{\varepsilon}) \varphi) = (\partial_1 \varphi \wedge \partial_2 \varphi) \cdot \varphi, \quad (92)$$

so that the energy contribution of the pressure load reduces to

$$P(\varepsilon)(v) = \int_{\Omega} \left[\varepsilon \pi \det(\partial_1 \varphi | \partial_2 \varphi | \frac{\partial_3 \varphi}{\varepsilon}) + \frac{\Delta \pi}{6} (\partial_1 \varphi \wedge \partial_2 \varphi) \cdot \varphi \right] dx. \quad (93)$$

Since we have assumed that $p > 3$, it is not difficult to see that Lemma 3 still holds true. In Propositions 5 and 7, we thus consider sequences $v(\varepsilon) \in V$ such that $v(\varepsilon) \rightharpoonup v$ with $v \in V_M$ and $(\partial_1 v(\varepsilon) | \partial_2 v(\varepsilon) | \varepsilon^{-1} \partial_3 v(\varepsilon))$ is bounded in $L^p(\Omega; M_3)$. For such sequences, we have

$$P(\varepsilon)(v(\varepsilon)) \rightarrow \frac{\Delta \pi}{3} \int_{\omega} (\partial_1(\psi + \bar{v}) \wedge \partial_2(\psi + \bar{v})) \cdot (\psi + \bar{v}) dx_1 dx_2. \quad (94)$$

Indeed, $\det(\partial_1 \varphi(\varepsilon) | \partial_2 \varphi(\varepsilon) | \varepsilon^{-1} \partial_3 \varphi(\varepsilon))$ is bounded in $L^{p/3}(\Omega)$, $\partial_1 \varphi(\varepsilon) \wedge \partial_2 \varphi(\varepsilon) \rightharpoonup \partial_1 \varphi \wedge \partial_2 \varphi$ in $L^{p/2}(\Omega; \mathbb{R}^3)$ by the weak continuity of null Lagrangians, see Ball [1977], Ball, Currie and Olver [1981], and $\varphi(\varepsilon) \rightarrow \varphi$ in $C^0(\bar{\Omega}; \mathbb{R}^3)$ by the Rellich-Kondrachov theorem (recall that Ω is Lipschitz). Hence, we obtain the Γ -limit

$$J^*(0)(v) = \begin{cases} 2 \int_{\omega} QW_0(x, (a_1 + \partial_1 \bar{v} | a_2 + \partial_2 \bar{v})) \sqrt{a} dx_1 dx_2 \\ \quad + \frac{\Delta \pi}{3} \int_{\omega} ((a_1 + \partial_1 \bar{v}) \wedge (a_2 + \partial_2 \bar{v})) \cdot (\psi + \bar{v}) dx_1 dx_2 & \text{if } v \in V_M, \\ +\infty & \text{otherwise,} \end{cases} \quad (95)$$

The limit energy expressed in terms of the deformations $\bar{\varphi} \in \bar{\Phi}$ then reads:

$$\bar{I}(0)(\bar{\varphi}) = 2 \int_{\omega} QW_0(x, \nabla \bar{\varphi}) \sqrt{a} dx_1 dx_2 + \frac{\Delta\pi}{3} \int_{\omega} (\partial_1 \bar{\varphi} \wedge \partial_2 \bar{\varphi}) \cdot \bar{\varphi} dx_1 dx_2. \quad (96)$$

As before, we may express this energy on the surface \tilde{S} itself. This yields:

$$\tilde{I}_{\tilde{S}}(\tilde{\varphi}) = 2 \int_{\tilde{S}} W_m(a_3(\tilde{x}), D\tilde{\varphi}(\tilde{x})) d\tilde{\sigma} + \frac{\Delta\pi}{3} \int_{\tilde{S}} (\text{cof } D\tilde{\varphi}(\tilde{x})a_3(\tilde{x})) \cdot \tilde{\varphi} d\tilde{\sigma}. \quad (97)$$

The term $\tilde{P}(\tilde{\varphi}) = \frac{\Delta\pi}{3} \int_{\tilde{S}} (\text{cof } D\tilde{\varphi}(\tilde{x})a_3(\tilde{x})) \cdot \tilde{\varphi} d\tilde{\sigma}$ corresponds to a pressure of amount $\Delta\pi$ applied on the deformed surface. Indeed, the Euler-Lagrange equations for the limit problem involve the term

$$D\tilde{P}(\tilde{\varphi})\tilde{v} = \Delta\pi \int_{\tilde{S}} (\text{cof } D\tilde{\varphi}(\tilde{x})a_3(\tilde{x})) \cdot \tilde{v} d\tilde{\sigma}$$

for all test functions \tilde{v} which vanish on $\partial\tilde{S}$. If we assume that the deformation $\tilde{\varphi}$ is smooth enough, this may be rewritten as an integral over the deformed surface

$$D\tilde{P}(\tilde{\varphi})\tilde{v} = \Delta\pi \int_{\tilde{\varphi}(\tilde{S})} n_{\tilde{\varphi}} \cdot (\tilde{v} \circ \tilde{\varphi}^{-1}) d\tilde{\sigma}_{\tilde{\varphi}}$$

where $n_{\tilde{\varphi}}(y)$ is the unit normal vector to $\tilde{\varphi}(\tilde{S})$ in the direction of $\partial_1 \bar{\varphi} \wedge \partial_2 \bar{\varphi}(\psi^{-1}(\tilde{\varphi}^{-1}(y)))$.

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