

Two Real Analysis Books that Changed My Life

Geometric Measure Theory by Herbert Federer

and

Introduction to Fourier Analysis on Euclidean Spaces

by E. M. Stein and Guido Weiss

Review Submitted by Steven G. Krantz

Those of us who were fortunate enough to have a solid, traditional mathematics education experienced, as the crucible of that training, a deep and difficult course in undergraduate analysis. The text typically was Rudin's *Principles of Mathematical Analysis* or another book of that ilk. It was easy to say, and often true, that anyone who could survive a year of Rudin was a real mathematician.

But such an education was a bit misleading. One came away from undergraduate analysis with the notion that the crux of the subject was analysis on the line. To be sure, one could pose and answer questions in higher dimensions; but \mathbb{R}^N was just an instance of a metric space. And, when properly viewed (with tools like the triangle inequality), metric spaces were much like the real line. End of discussion.

Unfortunately, this *Weltanschauung* is blissfully ignorant of the beautiful geometry and deep analytic structure of N -dimensional Euclidean space. Perhaps a simple example is in order. Suppose one wants to prove the (not terribly well-known) result that *any closed set* is the zero set of some C^∞ function. On the real line this is like shooting fish in a barrel. First, the complement of the closed set is open, and is thus the disjoint union of open intervals. On each such interval, one can erect a little bump that vanishes to infinite order at the endpoints. *Voila!* The union of these bumps is the graph of the C^∞ function that we seek.

This argument cannot work in higher dimensions. For it is no longer the case that every open set is the union of—of *what?* Certainly not of open cubes. In fact some real insight is required to discern the correct analog of the misleadingly simple result in \mathbb{R} . The *Whitney decomposition* of an open set does the trick. But it takes considerable technique to learn to use the Whitney decomposition. It is a slight over-simplification to say that this circle of ideas is what the book of Stein & Weiss (*Fourier Analysis*) is about.

What is marvelous about *Fourier Analysis* is that it shows that one can ask meaningful and exciting question about higher-dimensional Euclidean space, and then *one can answer them*. The tools that are developed along the way—the Fourier transform, spherical harmonics, interpolation of operators, multiplier problems, singular integrals, real-variable Hardy spaces—are among the most profound and original ideas in modern analysis. They do not simply describe the subject and provide tools to solve problems; in fact they *shape the subject*, and train us to ask new and deep questions.

The book of Federer teaches us the same lesson in a different argot. For Federer’s passion is geometry. And he teaches us, in delightful and compelling ways, that higher-dimensional geometry holds many mysteries that are inconceivable in one dimension. As a simple instance, consider this statement:

Let $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ be a simple, closed, C^2 curve with γ' never zero. Then there is an $\epsilon > 0$ such that each $x \in \mathbb{R}^2$ of distance less than ϵ from the curve has a *unique* nearest point on that curve. (*)

The proof of this result—fundamental to modern geometric analysis—is an easy exercise with the implicit function theorem. It is important to know that the result generalizes to hypersurfaces in higher-dimensional spaces. *And it fails for $C^{2-\delta}$ curves and surfaces.*

These considerations led Federer to formulate the notion of a *set of positive reach*, i.e., a set that enjoys Property (*). This idea is one of the key notions in Federer’s seminal paper “Curvature measures”, and forms one of many central ideas in the book *Geometric Measure Theory (GMT)*. More generally, *GMT* develops the tools necessary to solve the higher dimensional Plateau problem, that is, the problem of proving existence and regularity of a minimal surface spanning a given closed curve or surface. The constructions, due to Federer and Fleming, are completely original and stunning in their power and versatility. These include the creation of the powerful theory of currents (differential forms with distribution coefficients). And they have, in the past forty years, traveled in exciting ways from the theory of minimal surfaces and exerted a profound influence on geometry, complex analysis of several variables, partial differential equations, and many other parts of mathematics.

The thing that I enjoyed most about reading *GMT* is that Federer spends the first chapter re-inventing, from the ground floor, the subject of real analysis. Federer in effect reviews, in his own language and with his own emphases, my entire undergraduate and graduate career in analysis. Reading this material, I felt that I now understood what real analysis was all about—how it was structured and how it cohered. In particular, I now understood the *geometry* of space. I finally felt that I could use it as a tool. And watching a master like Federer made the process both credible and compelling.

Of course I could name a few dozen books that have exerted a powerful influence over my mathematical career. But I will always cherish these two as the ones that made me feel that I could be a mathematician, participate in the joy of discovery and creativity, and make a constructive and worthwhile contribution to the subject. The most important thing that we do as teachers is to act as role models for our students. Federer and Stein & Weiss have been role models for an entire generation of mathematicians.