Enumerating and characterizing real and complex singularities of curves and surfaces

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GAIA II— *Intersection algorithms for geometry based IT-applications using approximate algebraic methods*

- SINTEF (Norway)
- Johannes Kepler University (Austria)
- University of Nice–Sophia Antipolis (France)
- University of Cantabria (Spain)
- think3 (Italy and France)
- University of Oslo (Norway)

http://www.sintef.no/static/AM/gaiatwo/

Main objective of GAIA II is

- to improve intersection algorithms by integrating knowledge from real and classical algebraic geometry in CAD,
- and to span the whole chain from basic research to industrial prototyping.

There will be a workshop in cooperation with EAGER:

Algebraic Geometry and Geometric Modeling
September 27-29, 2004 in Nice, France

http://www-sop.inria.fr/galaad/conf/04aggm/

and another one at **CMA**, University of Oslo, in the fall of 2005.

1. Plane curves

2. Surfaces in 3-space

3. Veronese surfaces

4. Segre surfaces

5. Monoids

1. Plane curves

Let $C \subset \mathbb{C}P^2$ be a plane algebraic curve of degree n and class $n^\vee$.

Plücker formulas for curves with only nodes and ordinary cusps:

$$n^\vee = n(n-1) - 2\delta - 3\kappa$$

$$i = 3n(n-2) - 6\delta - 8\kappa$$

$$\Rightarrow 3(n^\vee - n) = i - \kappa$$

$\delta =$ the number of nodes
$\kappa =$ the number of cusps
$i =$ the number of inflection points

Kleins formula:

$$n^\vee - n = 2(\tau' - \delta') + i_r - \kappa_r$$

τ' = the number of real lines tangent to C at a pair of distinct conjugate points
δ' = the number of isolated real points (acnodes)
i_r = the number of real inflection points, and κ_r the number of real cusps

Corollary. If C is non-singular, then at most one third of its inflection points are real.

Proof. $i_r \leq 2\tau' + i_r = n^\vee - n = \frac{1}{3}i$

The dual curve $C^\vee \subset \mathbb{C}P^{2^\vee}$ has degree $n^\vee$ and same genus g as C. If κ (resp. i) denotes the ramification index of C (resp. $C^\vee$), then there are other Plücker formulas:

$$n^\vee = 2n + 2g - 2 - \kappa$$

$$n = 2n^\vee + 2g - 2 - i$$

$$\Rightarrow 2\kappa + i = 3(n + 2g - 2)$$

These formulas generalize to curves in higher-dimensional space.

For $g = 0$, if $\kappa = \kappa_r$ and $i = i_r$, the curve is called *maximally inflected* (Kharlamov–Sottile, related to the B. Shapiro–M. Shapiro conjecture; also recent work B. Osserman for higher genus).

The Klein–Schuh formula (Viro, Wall):

$$n^\vee - n = \sum\Big(m_{p^\vee}(C^\vee) - m_p(C)\Big)$$

where the sum is over pairs of points $(p, p^\vee) \in \widehat{C}(\mathbb{R})$ such that p is singular on C or $p^\vee$ is singular on $C^\vee$, and $\widehat{C}$ is the graph of the dual map (the conormal variety).

Can use Klein or Klein–Schuh to

- bound number of isolated points on a max inflected curve (K-S)

- prove real version of "the only uninflected curves are the rational normal curves" in odd-dimensional projective space (Huisman).

2. Surfaces in 3-space

Let $X \subset \mathbb{C}P^3$ be an algebraic surface. The numerical characters

$\mu_0 = n =$ degree of X
$\mu_1 =$ the rank, equal to the class of $X \cap H$
$\mu_2 =$ the class of $X = n^\vee$, the degree of $X^\vee$

can all be expressed in terms of characters of the strata of the singular locus of X. If X is a general projection of a smooth surface, then its singularities are: a nodal curve of degree m with a finite number t of triple points and a finite number ν_2 of pinch points (crosscaps, or umbrellas in the real case):

$$\mu_1 = n(n-1) - 2m$$

$$n^\vee = n(n-1)^2 - (3n-4)m + 3t - 2\nu_2$$

Trouble: the dual surface will usually have worse singularities. But μ_1 is also the rank of $X^\vee$, and the class of $X^\vee$ is the degree of X. Hence, without assumptions on the singularities on X and $X^\vee$:

$\mu_1 = 2n + 2g - 2 - \kappa$

$\mu_1 = 2n^\vee + 2g^\vee - 2 - \kappa^\vee$

$\Rightarrow 2n - e = 2n^\vee - e^\vee$ (Viro–Plücker)

g (resp. $g^\vee$) = genus of a plane section of X (resp. $X^\vee$)
κ (resp. $\kappa^\vee$) = degree of the "cuspidal edge" of X (resp. $X^\vee$)
$e := 2 - 2g + \kappa$ (resp. $e^\vee := 2 - 2g^\vee + \kappa^\vee$)

The real case

Let Eu_X denote the constructible function given by

$\text{Eu}_X(x) =$ the local Euler obstruction of X at x.

The local Euler obstruction has both an algebraic definition (in terms of the Nash blowup and Chern classes) and a topological definition.

Note that $\text{Eu}_X(x) = 1$ if $x \in X$ is nonsingular, but that the converse does not hold: if $x \in X$ is a pinch point, then also $\text{Eu}_X(x) = 1$.

Then we have Viro's formula:

$$\chi(\mathbb{R}P^{3\vee}, \text{Eu}_{X^\vee}) = \chi(\mathbb{R}P^3, \text{Eu}_X)$$

3. Veronese (a.k.a. triangular) surfaces

The Veronese embedding of degree d is the map

$$v_d : \mathbb{C}P^2 \to \mathbb{C}P^{\binom{d+2}{2}-1},$$

$$v_d(t_0 : t_1 : t_2) = (t_0^{i_0} t_1^{i_1} t_2^{i_2})_{i_0+i_1+i_2=d}$$

or, in affine coordinates: $(x, y) \mapsto (x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \ldots)$.

Any rational surface in $\mathbb{C}P^3$ that can be parameterized by polynomials of degree d is a projection of the Veronese embedding.

In the complex case a general projection has a double curve of degree m, t triple points, and ν_2 pinch points, given by

$$\begin{aligned} m &= d(d-1)(d^2+d-3)/2 \\ t &= \tfrac{1}{6}(d^6 - 12d^4 + 9d^3 + 44d^2 - 72d + 30) \\ \nu_2 &= 6(d-1)^2 \end{aligned}$$

Example: the case $d = 2$.

(i) If the projection is general, we have

$n = 4$, $m = 3$, $t = 1$, $\nu_2 = 6$, $\mu_1 = 6$ and $n^\vee = 3$.

In fact, the nodal curve is the union of three lines meeting in a non-planar triple point, and each of the lines contain two pinch points.

If the projection is not general enough, there are other possibilities:

(ii) the singular curve is the union of two intersecting lines, one of these is an ordinary nodal line on X and the other is a "tacnodal" line (meaning that a plane intersects X in a curve with a tacnode, or A_3 singularity);

(iii) Γ is a "higher order tacnodal" line on X, meaning a plane intersection has an A_5 singularity at the intersection with Γ.

(Plane curve singularities can be classified according to their resolution graph, or Enriques diagram.)

A general plane section $C = X \cap H$ is a plane rational quartic curve with

(i) three nodes (3 A_1)
(ii) one node and one tacnode ($A_1 + A_3$)
(iii) one higher tacnode (A_5).

In all cases, the rank of X is $\mu_1 = 6$, and its class is $\mu_2 = 3$.

So the dual surface $X^\vee \subset \mathbb{C}P^{3^\vee}$ is a cubic surface, of rank equal to the rank of X, namely 6, and of class 4, equal to the degree of X. From this, it follows that the cubic surface $X^\vee$ must be singular (otherwise it would have class 12), but that it cannot have a one-dimensional singular locus, since the rank is the maximum possible for a cubic.

Case (i): $X^\vee$ is equal to a cubic surface with 4 nodes and containing 9 lines.

Cases (ii) and (iii): the dual surface of $X^\vee$ is X, so we can use the classification of cubic surfaces: there are in all five types of cubic surfaces that have a dual surface of degree 4, so need only find out which of these types occur.

Can also use Singular to compute the equation of the dual of a typical surface in each of the cases (ii) and (iii).

Case (ii): We may assume the surface has equation

$$w^4 - 2w^2x(y+z) + x^2(x-z)^2 = 0.$$

The dual surface is the cubic

$$4stu - v^2(s+t) = 0.$$

One can show that this cubic surface has two A_1 singularities and one A_3 singularity, and it contains five lines.

Case (iii): We may assume the surface has equation

$$(wy - z^2)^2 - xy^3 = 0.$$

The dual surface is the cubic

$$s^2u - 4tuv + v^3 = 0.$$

This cubic surface has one A_1 and one A_5 singularity, and it contains two lines.

Note that the deformations (iii) $\to$ (ii) $\to$ (i), obtained by changing the center of projection, correspond to deformations of the dual surfaces, which fits with the deformations of simple singularities

$$A_1A_5 \to 2A_1A_3 \to 4A_1$$

corresponding to removing each time one (inner) vertex of the Dynkin diagram and its adjoining edges (cf. Looijenga, Bruce–Wall).

One of the real versions in case (i) of this example is Steiner's Roman surface:

$$x^2y^2 + x^2z^2 + y^2z^2 - xyz = 0$$

The three nodal lines are real, and each contains two real pinch-points.

There are two other different real types in case (i), depending on the reality of the components of the nodal curve and of the pinch points:

$x^2y^2 - x^2z^2 + y^2z^2 - xyz = 0$ and $x(y^2 + z^2) - x^2z^2 - (y^2 + z^2)^2 = 0$

The first has three real nodal lines, but only one contains two real pinch points. The other has only one real nodal line, with two real pinch points.

There are two different real types in case (ii):

$$y^2z - x^2z^2 + y^2 = 0$$

has one real nodal line with two real pinch points, and one real tacnodal line.

$$(x^2 - yz - z)^2 - 4yz^2 = 0$$

has one real nodal line with two real pinch points, one of which coincides with the triple point, and one tacnodal line.

In case (iii) there is one type:

$$(xz + y^2)^2 - z^3 = 0$$

with one real higher tacnodal line, a real triple point, and no real pinch points.

4. Segre (a.k.a. tensor product) surfaces

The Segre embedding of bidegree (a,b) is the map

$$\sigma_{a,b} : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^{(a+1)(b+1)-1},$$

$$\sigma_{a,b}((s_0 : s_1),(t_0 : t_1)) = (s_0^{i_0} s_1^{i_1} t_0^{j_0} t_1^{j_1})_{i_0+i_1=a, j_0+j_1=b}$$

Or, in affine coordinates:

$$(s,t) \mapsto (s,t,st,s^2,t^2,s^2t,st^2,s^2t^2,\ldots)$$

A general projection has a double curve of degree m, t triple points, and ν_2 pinch points, given by

$$\begin{aligned} m &= a^2b^2 - 2ab + (a+b)/2 \\ t &= 4ab(a^2b^2+11)/3 - 8a^2b^2 + 2ab(a+b) - 8(a+b) + 4 \\ \nu_2 &= 12ab - 8(a+b) + 4 \end{aligned}$$

Example 1. The biquadric surface

The projection of the Segre embedding with $a = b = 2$ has degree 8. The double curve has degree 10, there are 20 triple points and 20 pinch points.

Example 2. The bicubic surface

The projection of the Segre embedding with $a = b = 3$ has degree 18. The double curve has degree 66, there are 520 triple points and 64 pinch points.

We can look at special patches of bicubic surfaces:

$\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$, $\varphi(s,t) = (f_1(s)g(t), f_2(s)g(t), h(t))$

where f_1, f_2, g, h are cubic polynomials.

5. Monoid surfaces

A monoid surface $X \subset \mathbb{P}^3$ is a surface of degree d with a singular point of multiplicity $d-1$. If the point is $P = (0;0;0;1)$, then X has implicit equation of the form

$$x_3 f_{d-1}(x_0, x_1, x_2) - f_d(x_0, x_1, x_2) = 0,$$

and a rational parameterization

$$(x_0 : x_1 : x_2) \mapsto \left(x_0 : x_1 : x_2 : \frac{f_d}{f_{d-1}}\right).$$

The graph of this rational map is the surface obtained by blowing up the plane in the base points, i.e., the points of intersection of the two plane curves defined by $f_{d-1} = 0$ and $f_d = 0$. The type of the monoid singularity depends on the curve $f_{d-1} = 0$ and how it intersects the curve $f_d = 0$.

How to classify isolated surface singularities?

By numbers:

- the multiplicity
- the Milnor number
- the Milnor number of a plane section

Or by the resolution graph or its dual graph or its Dynkin diagram (e.g. A_k, D_k, E_k), or by its "normal form."

The Milnor number has both an algebraic and a topological definition; and it is equal to the dimension of the versal unfolding of the singularity, and equal to the number of "vanishing cycles."

Example.

Let $T[i, j, k]$ denote the singularity that has normal form

$x^i + y^j + z^k + xyz$, where $i, j, k \geq 3$.

The Milnor number of this singularity is $\mu = i + j + k - 1$.
The simplest of these is $P_8 = T[3, 3, 3]$.

Example: Quartic monoids

If $f_3 = 0$ is nonsingular, the (complex) type of the monoid singularity is $P_8 = T[3,3,3]$. There are two different associated real singularity types, depending on the connectedness of $f_3 = 0$.

Set $o := (0 : 0 : 1) \in \mathbb{C}P^2$, and let $I_o(f_3, f_4)$ denote the intersection multiplicity of the two curves in o. The table that follows is one of many that were made by Magnus Løberg (work in progress), using Singular's "quickclass" command.

If $f_3 = 0$ has a node:

f_3	f_4	$I_o(f_3, f_4)$	μ	Normal form
$(y^2 - x^2)z - x^3$	z^4	0	9	$T[3,3,4]$
$-xyz + x^3 + y^3$	$(x + y)z^3$	2	10	$T[3,3,5]$
$-xyz + 2x^3 + y^3$	$(zy - x^2)z^2$	3	11	$T[3,3,6]$
$-xyz + x^3 + y^3 + x^2y$	$(zy - x^2)z^2$	4	12	$T[3,3,7]$
$-xyz + x^3 + y^3 + xy^2$	$(zy - x^2)z^2$	5	13	$T[3,3,8]$
$-xyz + x^3 + y^3$	$(zy - x^2)z^2$	6	14	$T[3,3,9]$
$xyz + x^3 - x^2y + y^3$	$yz^3 - x^2z^2$ $+x^3z - x^4$	7	15	$T[3,3,10]$
$-xyz + x^3 + x^2y$ $+bxy^2 + (-3b - 1)y^3$	$yz^3 - x^2z^2$ $-x^3z - (b + 1)x^4$	8	16	$T[3,3,11]$

(In the last row, b satisfies $2b^2 - 6b - 3 = 0$.)