Editorial comment. The proposers remark that the identity is the key to proving an infinite class of "Apéry-like" series acceleration formulae for values of the Riemann zeta function.

Also solved by R. Chapman (U. K.), J. Grivaux (France), H. Wilf, and the proposers.

A Quick Inequality

11189 [2005, 929]. *Proposed by Lajos Csete, Markotabődőge, Hungary.* Let $a_1, \dots, a_n$ be positive real numbers, let $a_{n+1} = a_1$, and let p be a real number greater than 1. Prove that

$$\sum_{k=1}^{n} \frac{a_k^p}{a_k + a_{k+1}} \geq \frac{1}{2} p \sum_{k=1}^{n} a_k - \frac{p-1}{2^{p/(p-1)}} \sum_{k=1}^{n} (a_k + a_{k+1})^{1/(p-1)}.$$

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. By moving everything to the left side and combining the three sums, we can write the inequality to be proven as

$$\sum_{k=1}^{n} \frac{p}{2b_k} \left[\frac{a_k^p}{p} + \frac{b_k^q}{q} - a_k b_k \right] \geq 0,$$

where $b_k = (a_k + a_{k+1})/2$ and $q = p/(p-1)$. Therefore, it is enough to show that the expression between square brackets is nonegative. This is true since, for any fixed b, the function

$$f(x) = \frac{x^p}{p} + \frac{b^q}{q} - xb, \qquad x > 0,$$

has a global minimum at $x = b^{1/(p-1)}$, where it has value 0.

Also solved by R. Bagby, O. Bagdasar (Romania), R. Chapman (U. K.), P. P. Dályay (Hungary), G. Kiss (Hungary), R. Stong, L. Zhou, Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), Microsoft Research Problems Group, and the proposer.

A Polynomial and Its Zeros

11190 [2005, 929]. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain, and Pantelion George Popescu, Bucharest, Romania.* Suppose that A is a polynomial of degree n with $n \geq 2$, A has n distinct zeros $z_1, \dots, z_n$, and t is a complex number such that no ratio of two zeros of A is equal to t. Prove that

$$\sum_{k=1}^{n} z_k \left[\left(\frac{1}{A(tz_k)} + \frac{1}{t^2 A(z_k/t)} \right) \prod_{j \neq k} \frac{1}{(z_k - z_j)} \right] = 0.$$

Solution I by Julien Grivaux (student), Université Pierre et Marie Curie, Paris, France. Without loss of generality we may take A monic, so that $A(z) = \prod_{k=1}^{n} (z - z_k)$ with $n \geq 2$, and by continuity we may assume $t \neq 0$. Let $R(z) = z/\big(A(z)A(tz)\big)$. Since $t \neq 0$ and $\deg A \geq 2$, the residue at infinity of R is zero. We compute now the residues of R at the poles $z_1, \dots, z_n, z_1/t, \dots, z_n/t$, where all of these poles are distinct and of order one. It follows that

$$\operatorname*{Res}_{z_k} R = \frac{z_k}{A'(z_k)A(tz_k)} = \frac{z_k}{A(tz_k)} \prod_{j\neq k} \frac{1}{z_k - z_j}$$

$$\operatorname*{Res}_{z_k/t} R = \frac{z_k/t}{tA'(z_k)A(z_k/t)} = \frac{z_k}{t^2 A(z_k/t)} \prod_{j\neq k} \frac{1}{z_k - z_j}.$$

Since $\sum_{\alpha\in\mathbb{C}\cup\{\infty\}} \operatorname{Res}_\alpha R = 0$, the required result follows.

Solution II by C. J. Hillar, Texas A&M University, College Station, TX. As in the first solution, we take A to be monic. Note that it is enough to prove that the stated equality holds over the ring $\mathbb{Q}(z_1, \dots, z_n, t)$ in which $z_1, \dots, z_n, t$ are algebraically independent indeterminates, by the substitution principle. Our first step is to observe that

$$\frac{1}{A(tz_k)} + \frac{1}{t^2 A(z_k/t)} = \prod_{j=1}^{n} \frac{1}{(tz_k - z_j)} + t^{-2} \prod_{j=1}^{n} \frac{1}{(z_k/t - z_j)}$$
$$= \frac{1}{z_k(t-1)} \left[\prod_{j\neq k} \frac{1}{(tz_k - z_j)} - t^{-1} \prod_{j\neq k} \frac{1}{(z_k/t - z_j)} \right].$$

Therefore, it is enough to show:

$$\sum_{k=1}^{n} \prod_{j\neq k} \frac{1}{(z_k t - z_j)(z_k - z_j)} + (-t)^{n-2} \sum_{k=1}^{n} \prod_{j\neq k} \frac{1}{(z_j t - z_k)(z_k - z_j)} = 0. \qquad (1)$$

Multiply the left-hand side of (1) by $\prod_{i\neq j}(z_i t - z_j)$ to produce, since $n \geq 2$, the following polynomial in t:

$$F(t) = \sum_{k=1}^{n} \frac{\prod_{i\neq j,k}(z_i t - z_j)}{\prod_{j\neq k}(z_k - z_j)} + (-t)^{n-2} \sum_{k=1}^{n} \frac{\prod_{j\neq k,i}(z_i t - z_j)}{\prod_{j\neq k}(z_k - z_j)}. \qquad (2)$$

This polynomial has degree at most $n(n-1) - (n-1) + (n-2)$, which simplifies to $n(n-1) - 1$. Therefore, if we can prove that it has $n(n-1)$ distinct zeros, then it must be the zero polynomial, and that will prove (1). We may verify that $t_{r,s} = z_r/z_s$ for $r \neq s$ are $n(n-1)$ zeros of $F(t)$, and they are distinct since $z_1, \dots, z_n$ are algebraically independent.

For notational simplicity, we work out the case in which $r = 1$ and $s = 2$. The general situation is exactly the same. Upon substitution of $t = t_{1,2} = z_1/z_2$, each term in (2) vanishes except for the following two:

$$\frac{\prod_{i\neq j,2}(z_i t_{1,2} - z_j)}{\prod_{j\neq 2}(z_2 - z_j)} + (-t_{1,2})^{n-2} \frac{\prod_{j\neq 1,i}(z_i t_{1,2} - z_j)}{\prod_{j\neq 1}(z_1 - z_j)}.$$

Factoring out common products, we are reduced to proving that the following expansion is zero:

$$\begin{aligned}
&\frac{\prod_{i>2}(z_i t_{1,2} - z_1)}{\prod_{j>2}(z_2 - z_j)} - (-t_{1,2})^{n-2}\frac{\prod_{j>2}(z_2 t_{1,2} - z_j)}{\prod_{j>2}(z_1 - z_j)} \\
&= \frac{\prod_{i>2}(z_i z_1/z_2 - z_1)}{\prod_{j>2}(z_2 - z_j)} - (-\frac{z_1}{z_2})^{n-2}\frac{\prod_{j>2}(z_1 - z_j)}{\prod_{j>2}(z_1 - z_j)} \\
&= \left(-\frac{z_1}{z_2}\right)^{n-2}\frac{\prod_{i>2}(z_2 - z_i)}{\prod_{j>2}(z_2 - z_j)} - \left(-\frac{z_1}{z_2}\right)^{n-2}.
\end{aligned}$$

Since this last expression is indeed zero, the required result follows.

Also solved by M. R. Avidon, D. Beckwith, R. Chapman (U. K.), K. Dale (Norway), D. Donini (Italy), G. Keselman, J. H. Lindsey II, O. P. Lossers (Netherlands), A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), E. I. Verriest, Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

A Cyclic AM–GM Inequality

11193 [2005, 930]. *Proposed by Marian Tetiva, Bârlad, Romania.* Let $a_1, \ldots, a_n$ be positive real numbers. Let $a_{n+1} = a_1$. Prove that

$$\sum_{k=1}^{n}\left(\frac{a_k}{a_{k+1}}\right)^{n-1} \geq -n + 2\left(\sum_{k=1}^{n} a_k\right)\prod_{k=1}^{n} a_k^{-1/n}.$$

Solution by Koopa Koo, University of Washington, Seattle, WA. By the AM–GM inequality, and then by summing over k from 1 to n, we have

$$\left(\frac{a_k}{a_{k+1}}\right)^{n-1} + 1 \geq 2\left(\frac{a_k}{a_{k+1}}\right)^{\frac{n-1}{2}}, \qquad \sum_{k=1}^{n}\left(\frac{a_k}{a_{k+1}}\right)^{n-1} + n \geq 2\sum_{k=1}^{n}\left(\frac{a_k}{a_{k+1}}\right)^{\frac{n-1}{2}}.$$

To prove the original inequality, it therefore suffices to prove that

$$\sum_{k=1}^{n}\left(\frac{a_k}{a_{k+1}}\right)^{\frac{n-1}{2}} \geq \left(\sum_{k=1}^{n} a_k\right)\prod_{k=1}^{n} a_k^{-1/n}.$$

By the weighted AM–GM inequality, for each k we have (writing $a_{k+n} = a_k$)

$$\frac{(n-1)\left(\frac{a_k}{a_{k+1}}\right)^{\frac{n-1}{2}} + (n-2)\left(\frac{a_{k+1}}{a_{k+2}}\right)^{\frac{n-1}{2}} + \cdots + 2\left(\frac{a_{k+n-3}}{a_{k+n-2}}\right)^{\frac{n-1}{2}} + \left(\frac{a_{k+n-2}}{a_{k+n-1}}\right)^{\frac{n-1}{2}}}{(n-1) + (n-2) + \cdots + 2 + 1} \geq \left(\frac{a_k^n}{\prod_{s=1}^{n} a_s}\right)^{1/n}.$$

Summing over k from 1 to n gives

$$\sum_{k=1}^{n}\left(\frac{a_k}{a_{k+1}}\right)^{\frac{n-1}{2}} \geq \sum_{k=1}^{n}\left(\frac{a_k^n}{\prod_{s=1}^{n} a_s}\right)^{1/n} = \left(\sum_{k=1}^{n} a_k\right)\prod_{k=1}^{n} a_k^{-1/n}.$$

Also solved by O. Bagdasar (Romania), R. Chapman (U. K.), P. P. Dályay (Hungary), J. Grivaux (France), G. B. Passty, G. T. Prajitura, R. Stong, E. I. Verriest, L. Zhou, Microsoft Research Problems Group, and the proposer.