MATHEMATICAL MONTHLY PROBLEM SOLUTION

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ABSTRACT. [11028]. Proposed by Götz Trenkler. Let $P$ and $Q$ be Hermitian idempotent $n$-by-$n$ matrices with complex entries. That is, each is equal to its own square and its own conjugate transpose. Show that $PQ$ shares these properties if and only if the trace of $PQPQ$ is equal to the trace of $PQ$.

1. Solution

One direction is obvious. As for the converse, suppose that $\text{Tr}[PQPQ - PQ]$ is zero. Consider now $\text{Tr}[(QP - PQ)(QP - PQ)^*]$, which is zero if and only if $QP - PQ = 0$ (since $\text{Tr}[AA^*]$ is the sum of the squares of the absolute values of the entries of $A$).

An easy computation using the assumptions does indeed give us 0 for such a trace:

$\text{Tr}[(QP - PQ)(QP - PQ)^*] = \text{Tr}[QP^2Q - QPQP + PQPQ + P^2PQ]$  
$= \text{Tr}[QPQ] - \text{Tr}[QPQP] - \text{Tr}[PQPQ] + \text{Tr}[PQP]$  
$= 2\text{Tr}[PQ] - 2\text{Tr}[PQPQ]$  
$= 0$

by assumption and repeatedly using the fact that $\text{Tr}[AB] = \text{Tr}[BA]$. It follows, therefore, that

$QP - PQ = 0$

which solves the problem.

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