11077: THE ASYMPTOTIC BEHAVIOR OF A CERTAIN SUM

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Abstract. Let \( x_n = \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \frac{2^{n-i} - 1}{n-i} \). Find the asymptotic behavior of \( x_n \).

1. Solution

We prove that
\[
\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \frac{2^{n-i} - 1}{n-i} = (-1)^n \sum_{j=1}^{n} \frac{(-1)^j}{j}.
\]

This shows that \( x_n \) tends to \( (-1)^n \ln 2 \). First preprocess (by replacing \( n - i \) with \( j \)) the given equation so that we need only show:
\[
\sum_{j=1}^{n} (-1)^j \binom{n}{j} \frac{2^j - 1}{n-j} = \sum_{j=1}^{n} \frac{(-1)^j}{j}.
\]

Clearly, the relation holds for \( n = 1 \) as both sides are \(-1\). Writing \( y_n \) for the right-hand-side (and \( z_n \) for the left-hand-side), notice that \( y_n = y_{n-1} + (-1)^n/n \).

Since \( z_1 \) and \( y_1 \) agree, it suffices to verify that \( z_n \) satisfies the same recurrence as \( y_n \). Examine the difference
\[
z_{n+1} - z_n = \sum_{j=1}^{n+1} (-1)^j \binom{n+1}{j} \frac{2^j - 1}{n+1-j} \]
\[
= \frac{1}{n+1} \sum_{j=1}^{n+1} (-1)^j \binom{n+1}{j} (2^j - 1)
\]
\[
= \frac{1}{n+1} \left[ \sum_{j=1}^{n+1} (-1)^j \binom{n+1}{j} 2^j - \sum_{j=1}^{n+1} (-1)^j \binom{n+1}{j} \right]
\]
\[
= \frac{1}{n+1} \left[ (1-2)^{n+1} - 1 - ((1-1)^{n+1} - 1) \right]
\]
\[
= \frac{1}{n+1} (-1)^{n+1}.
\]

This completes the proof.

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