CHARACTERISTIC EQUATION PROBLEM SOLUTION

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Abstract. Proposed by Matthias Beck, Jesus DeLoera, Mike Develin, and Julian Pfeifle. Let \( d \) be a positive integer, \( t_1, \ldots, t_d \) be integers, and \( \lambda_1, \ldots, \lambda_d \) be real numbers. Prove that if \( \sum_{k=1}^{d} \lambda_k t_k^j \) is an integer for \( 1 \leq j < d \), then also \( \sum_{k=1}^{d} \lambda_k t_k^d \) is an integer.

1. Solution

The statement as published is false. Take for example \( d = 2, t_1 = 2, t_2 = -1, \lambda_1 = \sqrt{2}/2, \lambda_2 = \sqrt{2} \). Then, \( \sum_{k=1}^{2} \lambda_k t_k = 0 \) is an integer whereas \( \sum_{k=1}^{2} \lambda_k t_k^2 = 3\sqrt{2} \) is not. One fix (see below for another) is to add the additional assumption that \( \lambda_1 + \cdots + \lambda_d \) is also an integer, and with this in place, we argue as follows.

Let \( a_n = \sum_{k=1}^{d} \lambda_k t_k^n \) for \( n \geq 1 \), and define \( a_0 = \lambda_1 + \cdots + \lambda_d \). From the elementary theory of linear recurrences, this sequence satisfies:

\[
a_{n+d} = \sum_{i=1}^{d} (-1)^{i-1} e_i a_{n+d-i},
\]

in which \( e_i = e_i(t_1, \ldots, t_d) \) is the \( i \)-th elementary symmetric function evaluated at \( t_1, \ldots, t_d \). For example, with \( d = 2 \), this identity simply reads:

\[
\lambda_1 t_1^{n+2} + \lambda_2 t_2^{n+2} = (t_1 + t_2)(\lambda_1 t_1^{n+1} + \lambda_2 t_2^{n+1}) - (t_1 t_2)(\lambda_1 t_1^n + \lambda_2 t_2^n).
\]

From this identity and the assumptions, it follows easily that \( a_n \) is integral for all \( n \geq 0 \). 

Remark 1.1. We note that one could also “fix” the statement by stating instead:

Prove that if \( \sum_{k=1}^{d} \lambda_k t_k^j \) is an integer for \( 1 \leq j \leq d \), then also \( \sum_{k=1}^{d} \lambda_k t_k^{d+1} \) is an integer.

The recurrence above would also work in this case.

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