AMM11096: THE DETERMINANT AS A POLYNOMIAL IN THE TRACES OF POWERS

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Abstract. [11096] Proposed by Said Amghibech. Show that for each positive integer $n$ there exists a polynomial $P_n$ in $\mathbb{C}[x_1, \ldots, x_n]$ such that, for every $n$-by-$n$ matrix $A$ over $\mathbb{C}$, $\det A = P_n[\text{Tr} A, \text{Tr} A^2, \ldots, \text{Tr} A^n]$.

1. Solution

The question may be restated without reference to matrices as follows. Let $R = \mathbb{Q}[\alpha_1, \ldots, \alpha_n]$ be the polynomial ring in algebraically independent indeterminates $\alpha_1, \ldots, \alpha_n$. Let $s_m$ denote the $m$-th power sum $\sum_{i=1}^{n} \alpha_i^m$. Then, the problem is asking for a polynomial $h(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ such that

$$\prod_{i=1}^{n} \alpha_i = h(s_1, \ldots, s_n).$$

However, this trivially follows from the (well-known) fact that power sums generate the ring of symmetric functions over $\mathbb{Q}$ (a typical proof uses Newton’s identities).

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