MATHEMATICAL MONTHLY PROBLEM PROPOSAL

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ABSTRACT. [11422] Let $H = H^T$ be an n-by-n real symmetric matrix with distinct eigenvalues and let A be an $n \times n$ real matrix. For $i \in \mathbb{N}$, define matrices $H_{i+1} = [A, H_i] = AH_i - H_iA$ in which $H_0 = H$. Prove that if H_1 and H_2 are symmetric then A is a normal matrix (that is, $AA^T = A^TA$).

1. Solution

We will describe the characterization of A mentioned in the abstract. However, we will consider a slightly more general setup by assuming that $H = H^*$ is complex Hermitian and A is a complex matrix. Let $r_1, \ldots, r_k \in \mathbb{C}$ and suppose that $A_1, \ldots, A_k$ are skew-Hermitian matrices (of possibly different sizes). Then, any unitary similarity of a block diagonal matrix of the form,

$$S = \begin{bmatrix} A_1 + r_1 I_1 & & 0 \\ & \ddots & \\ 0 & & A_k + r_k I_k \end{bmatrix} \tag{1.1}$$

(in which I_j is the identity matrix of the same size as A_j) is called a *quasi skew-Hermitian* matrix. Notice that any matrix S as above is normal and, therefore, so is any quasi skew-Hermitian matrix. We may now state the solution to the problem.

Proposition 1.1. *Let H be an n-by-n Hermitian matrix with distinct eigenvalues and let $A \in M_n(\mathbb{C})$. Then, H_i is Hermitian for $i = 1, 2$ if and only if A is quasi skew-Hermitian.*

Proof. We first prove necessity. Since H is Hermitian, there exists a unitary matrix U such that $UHU^* = E = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ in which $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are the distinct eigenvalues of H. Since

$$UH_{i+1}U^* = UAU^*UH_iU^* - UH_iU^*UAU^*$$

is Hermitian if and only if H_{i+1} is Hermitian, it follows that we may work with E in place of $H_0 = H$ (this will also not change any quasi skew-Hermitian conclusions about A).

Let $A = (a_{ij})$ and suppose that $H_1 = AE - EA = (h_{ij})$ is Hermitian. A straightforward computation gives us that $h_{ij} = a_{ij}(\lambda_j - \lambda_i)$, and thus the condition $H_1^* = H_1$ amounts to $a_{ij}(\lambda_j - \lambda_i) = \overline{a_{ji}}(\lambda_i - \lambda_j)$. In particular, since the λ_i are distinct, it follows that for $i \neq j$,

$$a_{ij} = -\overline{a_{ji}}. \tag{1.2}$$

Let D denote the matrix, $\text{diag}(a_{11}, \ldots, a_{nn})$, and notice that $A^* = D + D^* - A$. Since $H_2 = AH_1 - H_1A$ is assumed to be Hermitian, we must have

$$\begin{aligned} AH_1 - H_1A &= H_1A^* - A^*H_1 \\ &= H_1(D + D^* - A) - (D + D^* - A)H_1 \\ &= H_1D + H_1D^* - H_1A - DH_1 + D^*H_1 - AH_1. \end{aligned} \tag{1.3}$$

Cancelling common terms, we arrive at the equation

$$D^*H_1 - H_1D^* = H_1D - DH_1.$$

Of course, this last expression just means that $(c_{ij}) = D^*H_1 - H_1D^*$ is Hermitian. As before, it is easy to see that

$$c_{ij} = h_{ij}(\overline{a_{ii}} - \overline{a_{jj}}) = a_{ij}(\lambda_j - \lambda_i)(\overline{a_{ii}} - \overline{a_{jj}}).$$

As a consequence of this and (1.2), we notice that for $i \neq j$ and $a_{ij} \neq 0$,

$$a_{jj} + \overline{a_{jj}} = a_{ii} + \overline{a_{ii}}.$$

In particular, it follows that

$$\text{whenever } i \neq j \text{ and } a_{ij} \neq 0, \text{ the real parts of } a_{ii} \text{ and } a_{jj} \text{ coincide.} \tag{1.4}$$

Let P be any transposition permutation matrix. It is straightforward to verify (see Figure 1 below) that if A has the property (1.4) above, then so does PAP^T.

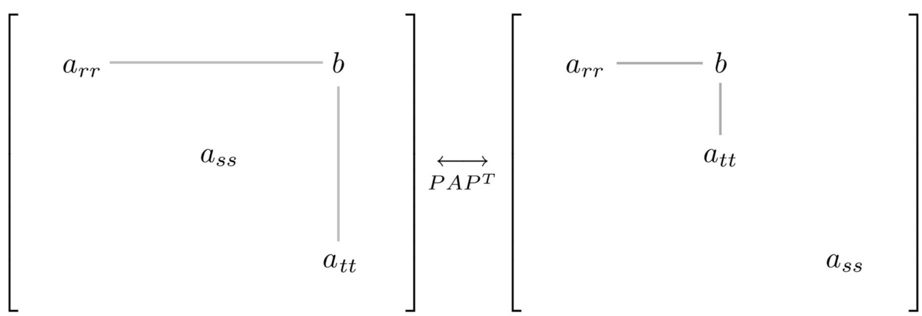

Figure 1. The Effect of a Transposition

Finally, let P be a permutation matrix such that the diagonal elements of $B = PAP^T$ are sorted by increasing real parts. Since property (1.4) is preserved, B must be a block diagonal matrix of the form

$$B = \begin{bmatrix} A_1 + r_1I_1 & & 0 \\ & \ddots & \\ 0 & & A_k + r_kI_k \end{bmatrix} \tag{1.5}$$

in which $A_1, \ldots, A_k$ are skew-Hermitian and $r_1, \ldots, r_k$ are the corresponding real parts of the diagonal entries of the blocks of B. It now follows that A is quasi skew-Hermitian as was to be proved. As the above arguments are reversible, sufficiency is also clear. This completes the proof. □

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