

$$p(t) = \text{tr}[(A+tB)^m] \in \mathbf{R}_+[t]$$

# Advances on the BMV Trace Conjecture

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# Motivation: Statistical Physics

In 1975, Bessis, Moussa, and Villani (BMV) introduced a positivity conjecture while studying partition functions of quantum mechanical systems.

Fix  $A, B$  to be  $n \times n$  positive semidefinite matrices (symmetric, nonnegative eigenvalues)

Define:  $f : [0, \infty) \rightarrow (0, \infty)$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

$$f(t) = \text{tr}[\exp(A - tB)]$$

Conjecture [BMV1]:  $f$  is completely monotone:

$$(-1)^m f^{(m)}(t) \geq 0, \quad t > 0, m = 0, 1, 2, \dots$$

**Example** ( $n = 1$ ):  $f(t) = e^{A-tB}$  is a scalar function

$$f^{(m)}(t) = (-B)^m f(t)$$

Generally,  $n > 1$ ,

$$f(t) = \text{tr}(I) + \text{tr}[(A-tB)] + \text{tr}[(A-tB)^2/2] + \dots$$

is very complicated (noncommutativity)

Lieb, Seiringer (2004) proved a local version is enough

**Conjecture [BMV2]**: For each  $m$ , the polynomial in  $t$

$$p(t) = \text{tr}[(A+tB)^m]$$

has nonnegative coefficients

**Remark**: again, this is trivial for  $n = 1$

**Example:** If  $m = 2$ , then conjecture BMV2 asserts

$$\text{tr}[(A+tB)^2] = \text{tr}[B^2] t^2 + \text{tr}[AB+BA] t + \text{tr}[A^2]$$

has nonnegative coefficients. Since  $A$  and  $B$  are positive semidefinite, only the second one,  $2\text{Tr}[AB]$ , poses any difficulty:

**Lemma:** If  $A, B$  are PSD, then  $AB$  has nonnegative eigenvalues.

In fact, this simple calculation proves BMV2 in the case  $m = 2$ , and similar tricks verify it up to  $m = 5$ .

However, no simple approach works for larger  $m$

# Many Attempts

- **variational** approaches [Le Couteur, H., Johnson]
- **hypergeometric** approaches [Drmota, Schachermayer, Teichmann]
- **free probability** approaches [Fannes, Petz]
- techniques using **matrix analysis** [H., Johnson, Miller, Hansen]
- **noncommutative sum of squares** approaches [Burgdorf, Haegele, Klep, Schweighofer, Landweber, Speer] that
- have been combined with the technology of **semidefinite programming** [Klep, Schweighofer].

Despite much work, little is known about the problem, and has remained unresolved except in special cases.

**Definition:** Focusing on individual coefficients, we define matrices

$$S_{m,k}(A,B) = [t^k] (A+tB)^m,$$

the sum of all length  $m$  words in  $A$  and  $B$  with  $k$   $B$ s.

$$S_{2,1}(A,B) = AB + BA$$

$$S_{3,2}(A,B) = ABB + BAB + BBA$$

Each  $S_{m,k}(A,B)$  is **Symmetric**, but the matrix need not be positive semidefinite (a natural first guess)

We refine BMV2 by fixing  $m$  and  $k$  :

**Conjecture [BMV3]:** For all positive semidefinite  $A$  and  $B$ , the matrix  $S_{m,k}(A,B)$  has nonnegative trace.

# Example: $m = 6, k = 3$

We focus on the case  $m = 6, k = 3$  to illustrate the general difficulty of conjecture BMV3. A simple computation gives

$$\text{tr}[S_{6,3}(A,B)] = 2\text{tr}[(AB)^3] + 6\text{tr}[A^3B^3] + 12\text{tr}[ABABBA]$$

The first two terms in the sum above are always nonnegative, but it turns out the the **third summand can be made negative** [2002 H., Johnson] although not easily so. More remarkably, picking random  $A$  and  $B$  fails to find such examples.

**Conclusion:** If the conjecture is true, almost certainly a proof must navigate a complicated sign cancellation

# Known Results

- When the size of the matrices is  $n = 2$ , the conjecture is well-known (any word  $W(A,B)$  has nonnegative eigenvalues)
- If  $AB = BA$  or  $m < 6$ , the conjectures are true.
- BMV1 is true in an “average” sense, the precise statement involving notions of free probability (matrix version of random variables) [Fannes, Petz]
- BMV2 is true when  $n = 3$  and  $A$  and  $B$  are of a special form [Drmota, et al]
- BMV3 is true for  $n = 3, m = 6, k = 3$ . [H., Johnson]
- BMV3 is true if either  $A$  or  $B$  have exactly 2 distinct e-values

# Recent Progress

It is difficult to get info about  $S_{m+1,k}$  from  $S_{m,k}$ ; yet

**Theorem 1 (Asymptotic Formulation):** If there are PSD  $A, B$  and a power  $m > 0$  showing BMV3 false, then there are PSD  $A, B$  making BMV3 false for power  $m+1$ .

**Corollary:** If conjecture BMV3 is false, then it is really false.

**Corollary:** If conjecture BMV2 is true for infinitely many  $m$ , then it is true for all  $m$ .

**Theorem 2 (Transfer Principal):**  $\text{Tr}[S_{m,k}(A, B)] \geq 0$  for all PSD  $A, B$  if and only if whenever  $S_{m,k}(A, B) \neq 0$ , it has at least one positive eigenvalue.

This last result essentially transfers the problem of showing that a **sum of eigenvalues** is nonnegative to that of showing **one of them is**.

In this regard, one is reminded of **Perron's Theorem** for nonnegative matrices. Fix PSD matrices  $A$  and  $B$ .

**Conjecture:** The matrix  $S_{m,k}(A,B)$  either has a positive eigenvalue or it is the zero matrix.

Finally, the following reduces the number of parameters in a computational approach by 2.

**Theorem 3 (Singular Reduction):** It is enough to prove conjecture BMV3 in the case of singular matrices.

These results inspire a program to prove the BMV conjecture:

**Program:** Try and find a “nice” (infinite) set of powers  $m$  for which BMV3 reduces to a sum of squares: For each  $m$  that one proves BMV3, one has BMV3 for all smaller  $m$ .

**Recent news:** Success has been made by Haegerle who found a simple proof for  $m = 7$  (and all  $n$  - a first nontrivial result for all sizes of matrices). An automatic corollary is that BMV3 is true for  $m = 6$

$$\text{tr}(S_{7,3}) = 7\text{tr}(AABBAAB) + 7\text{tr}[(ABA+AAB)B(ABA+BAA)]$$

Strange phenomenon though:

**Theorem [Haegele; Klep, Schweighofer]:** There is no sum of squares proof for  $m = 6$ .

This explains the difficulty of our previous work on the case  $m = 6, n = 3$ : To prove  $m = 6$  (for all  $n$ ), one must prove it for a higher  $m$  (in this case 7) and then descend downward. There is no other known proof of this difficult case.

**Initiates a race:** Prove it for a large  $m$ , and you automatically encompass all previous results

**Current records:** Klep and Schweighofer have a SOS proof for  $m = 13$  that used semidefinite programming

# The End

(of talk)