Journal of Symbolic Computation 74 (2016) 650–652

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Corrigendum

Corrigendum to "Finiteness theorems and algorithms for permutation invariant chains of Laurent lattice ideals" [J. Symb. Comput. 50 (March 2013) 314–334]

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ARTICLE INFO

Article history:
Received 21 May 2015
Accepted 16 July 2015
Available online 26 September 2015

We correct errors in the proof of Hillar and del Campo (2013, Theorem 19) and in the description of Hillar and del Campo (2013, Table 1). We also fix some typos and notational issues. We thank Thomas Kahle for pointing out the mistake in the table and Robert Krone for finding the gap in our original proof.

We begin with a comment to avoid confusion in the proof of Hillar and del Campo (2013, Proposition 13). Let $\preceq_{dlex}$ be the partial ordering of $[\mathbb{P}]^k$ (as defined in Hillar and del Campo, 2013, Equation (6)) induced by the degree lexicographic order $\leq_{dlex}$ and the action of the group $\mathfrak{S}_{\mathbb{P}}$. To prove Proposition 13, we used Higman's lemma (Hillar and del Campo, 2013, Lemma 14) to demonstrate that the order $\preceq_{dlex}$ is a well-partial-ordering of $[\mathbb{P}]^k$. In the argument, we considered the alphabet $\Sigma = \{0, 1, \ldots, k\}$ yielding the set Σ^* of finite sequences of elements of Σ. For $w = (w_1, \ldots, w_k) \in [\mathbb{P}]^k$, we used an additive notation to define $w^* \in \Sigma^*$ by

$$w_i^* := \sum_{w_j = i} j, \quad \text{for } i = 1, \ldots, n,$$

DOI of original article: http://dx.doi.org/10.1016/j.jsc.2012.06.006.
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http://dx.doi.org/10.1016/j.jsc.2015.09.002
0747-7171/

which denoted the sequence of elements of the set $\{j \mid w_j = i\}$. However, this formal sum notation above may cause confusion with integer addition. To avoid this ambiguity, one may consider Σ as the power set $2^{[k]}$ of all subsets of $[k] = \{1, \ldots, k\}$, and define $w^* \in \Sigma^*$ by $w_i^* := \{j \mid w_j = i\}$, for $i = 1, \ldots, n$.. Note also that $(3\ 4) \cdot (2, 3)_{<_{dlex}}$ in Example 10 is missing the element $(1, 1)$.

We now present a complete proof of the following result.

Theorem 1. *(See Theorem 19 in Hillar and del Campo, 2013.) Let A be a Noetherian commutative ring. For every $A[\mathfrak{S}_\mathbb{P}]$-submodule $B \subseteq A[\mathbb{P}]^k$, there exists a finite set $G \subseteq B$ such that*

$$f \in B \cap A[m]^k \iff \exists \sigma_1, \ldots, \sigma_\ell \in \mathfrak{S}_m;\ g_1, \ldots, g_\ell \in G;\ a_1, \ldots, a_\ell \in A \ \ with\ \ f = \sum_{i=1}^{\ell} a_i \sigma_i g_i.$$

In the original proof of Theorem 1, the proposed set G consisted of elements of the form $c_i b_j \in A[\mathbb{P}]^k$ with $c_i \in A$ and $b_j \in A[\mathbb{P}]^k$, where we assumed the latter to be monic for all $1 \le j \le |T|$. Unfortunately, these b_j are not necessarily monic, which makes the original proof incorrect. However, we exhibit a correct finite generating set G using the following elementary observation from Aschenbrenner and Hillar (2007, Lemma 3.13). Recall that a *well-founded* partially-ordered set is one with no infinite strictly decreasing sequences.

Lemma 2. *Let S be a well-partially-ordered set and T a well-founded partially-ordered set. Suppose that $\varphi: S \to T$ is decreasing: $s \le t \Rightarrow \varphi(s) \ge \varphi(t)$, for all $s, t \in S$. Then $\le_\varphi$ on S defined by*

$$s \le_\varphi t \quad :\iff \quad s \le t \ \wedge\ \varphi(s) = \varphi(t)$$

is a well-partial-ordering. □

Proof of Theorem 1. Let $\preceq_{dlex}$ be the well-partial-ordering of $[\mathbb{P}]^k$ induced by the total well-order $\le_{dlex}$. For $f \in A[\mathbb{P}]^k$, define the leading monomial $\mathrm{lm}(f)$ to be the largest nonzero element in $[\mathbb{P}]^k$ (with respect to $\le_{dlex}$) in the support of f. Also, set $\mathrm{lc}(f) \in A$ to be the coefficient of $\mathrm{lm}(f)$ in f.

Let T be the collection of all ideals of A (partially ordered $\preceq$ by reverse inclusion), and consider the following subcollection:

$$\mathrm{lc}(B, h) := \{0\} \cup \{\mathrm{lc}(f) \mid f \in B,\ \mathrm{lm}(f) = h\}.$$

Notice that $u \preceq_{dlex} v$ implies $\mathrm{lc}(B, u) \subseteq \mathrm{lc}(B, v)$ (so that $\mathrm{lc}(B, u) \succeq \mathrm{lc}(B, v)$). Using Lemma 2 with $S = ([\mathbb{P}]^k, \preceq_{dlex})$ and φ the decreasing map $\varphi(u) = \mathrm{lc}(B, u)$, we have that $\preceq_\varphi$ is a well-partial order.

Recall that a *final segment* of the partial order $\preceq_{dlex}$ is a set $F \subseteq [\mathbb{P}]^k$ such that $u \in F$ and $u \preceq_{dlex} v$ implies that $v \in F$. A well-known characterization of well-partial-orderings (see e.g. Kruskal, 1972) is that final segments are finitely generated. Thus, applied to the final segment $F = S$ of the well-quasi-ordering $\preceq_\varphi$, we obtain finitely many $w_1, \ldots, w_m \in [\mathbb{P}]^k$ with the following property: for every $w \in [\mathbb{P}]^k$ there exists $i \in \{1, \ldots, m\}$ such that $w_i \preceq w$ and $\mathrm{lc}(B, w_i) = \mathrm{lc}(B, w)$. Using Noetherianity of A, for each i now choose finitely many nonzero elements $g_{i1}, \ldots, g_{in_i}$ of B, each with leading monomial w_i, whose leading coefficients generate the ideal $\mathrm{lc}(B, w_i)$ of A.

We claim that the following subset of B fulfills the requirements of the theorem statement:

$$G := \{g_{ij} : 1 \le i \le m,\ 1 \le j \le n_i\}.$$

To prove this, let $f \in B \cap A[m]^k$. Then, $\mathrm{lm}(g) \preceq_{dlex} \mathrm{lm}(f)$ for some $g \in G$ with witness $\sigma_1 \in \mathfrak{S}_m$ (by Hillar and del Campo, 2013, Lemma 18). From the above construction, there are $a_1, \ldots, a_r \in A$ such that

$$f_1 := f - \sum_{i=1}^{r} a_i \sigma_1 g \in B$$

has a strictly smaller (with respect to $\le_{dlex}$) leading term than f. Continuing in this manner, we can produce a sequence $f_1, f_2, \ldots$ of elements in B such that

$$\cdots \le_{dlex} \mathrm{lm}(f_2) \le_{dlex} \mathrm{lm}(f_1) \le_{dlex} \mathrm{lm}(f).$$

Table 1
Degree-complexity of the toric ideal I_n defined by $\mathbf{y}^\alpha$.

$\alpha \setminus n$	3	4	5	6	7	8
(1, 1)	0	2	3	3	3	3
(2, 1)	3	4	4	4	4	4
(3, 1)	5	5	6	6	6	6
(4, 1)	7	7	7	7	7	7
(5, 1)	9	9	9	9	9	9
(6, 1)	11	11	11	11	11	11
(7, 1)	13	13	13	13	13	13
(8, 1)	15	15	15	15	15	–
(3, 2)	5	6	6	6	6	6
(4, 2)	3	4	4	4	4	4
(5, 2)	8	8	9	9	9	9
(6, 2)	5	5	6	6	6	6
(7, 2)	12	12	12	12	12	–
(1, 3, 2)	3	5	7	8	8	–
(4, 3, 2)	3	5	7	11	11	–

Since $\leq_{dlex}$ is a well-ordering, it follows that $f_p = 0$ for some $p \in \mathbb{P}$ which gives an expansion for f as in the statement of the theorem. $\square$

Next, we discuss our mistake in Hillar and del Campo (2013, Table 1). The original motivation for displaying this table was to suggest the stabilization (as n grows) of the maximal degree of a generating set for chains of toric ideals induced by monomials y^α (introduced in Hillar and del Campo, 2013, Section 4) as evidence for the stabilization of the chain of ideals. Recall that for a homogeneous ideal the *degree-complexity* of a reduced Gröbner basis is the maximum degree of its elements (Bayer and Mumford, 1993). In addition, one defines the *Markov degree* as the minimal value for the maximal degree of a generating set (i.e. a *Markov basis*). Notice that the degree-complexity is an upper bound for the Markov degree. Therefore, stabilization of the degree-complexity for a reduced Gröbner basis implies stabilization of the Markov degree.

One way to compute the degree-complexity of a homogeneous toric ideal (as in our case) is to call the `groebner` command in `4ti2` (4ti2 team) to obtain a reduced Gröbner basis (with respect to some internal term order) and compute the maximal degree of this generating set. In our code we extracted a minimal generating set from the reduced Gröbner basis (which is equivalent to calling the `markov` command in `4ti2`; hence, the mistake of our computation) and then computed its maximal degree. Therefore, the information displayed in Hillar and del Campo (2013, Table 1) are upper bounds for the Markov degree instead of the degree-complexity for the Gröbner bases. This information is useful, as the table suggests the stabilization of the Markov-degree as n grows. In Table 1 above, we present the correct summary of our degree-complexity calculations.

We conclude by pointing the reader to some recent references. The paper Kahle et al. (2014) contains bounds for the Markov degree and a construction of an equivariant Markov basis for chains of toric ideals induced by $\mathbf{y}^\alpha$ for $\alpha \in \mathbb{Z}^2$ with $\gcd(\alpha_1, \alpha_2) = 1$. Additionally, the preprint Draisma et al. (2013) contains proofs settling in the affirmative several stabilization questions involving (non-Laurent) toric ideals in Aschenbrenner and Hillar (2007), Hillar and del Campo (2013).

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