Linear Algebra and its Applications 472 (2015) 135–141

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A Hadamard-type lower bound for symmetric diagonally dominant positive matrices

Christopher J. Hillar [a,*,1], Andre Wibisono [b,*]

[a] Redwood Center for Theoretical Neuroscience, University of California, Berkeley, United States
[b] Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, United States

ARTICLE INFO

Article history:
Received 8 April 2014
Accepted 8 January 2015
Available online xxxx
Submitted by B. Lemmens

MSC:
15B48
15A15
15A45

Keywords:
Diagonally dominant
Positive matrices
Determinantal inequality
Hadamard's inequality

ABSTRACT

We prove a new lower-bound form of Hadamard's inequality for the determinant of diagonally dominant positive matrices.

* Corresponding authors.
E-mail addresses: chillar@berkeley.edu (C.J. Hillar), wibisono@eecs.berkeley.edu (A. Wibisono).

[1] Partially supported by NSF grant IIS-0917342 and an NSF All-Institutes Postdoctoral Fellowship administered by the Mathematical Sciences Research Institute through its core grant DMS-0441170.

http://dx.doi.org/10.1016/j.laa.2015.01.037

1. Introduction

An $n \times n$ real matrix J is *diagonally dominant* if

$$\Delta_i(J) := |J_{ii}| - \sum_{j \neq i} |J_{ij}| \geq 0, \quad \text{for } i = 1, \ldots, n.$$

A particularly interesting case is when $\Delta_i(J) = 0$ for all i; we call such matrices *diagonally balanced*. Irreducible, diagonally dominant matrices are always invertible, and such matrices arise often in theory and applications. In this Note we study bounds on the determinant of symmetric diagonally dominant matrices that have positive entries. These matrices are always positive definite (e.g., by Lemma 2.1).

It is classical that the determinant of a positive semidefinite matrix A is bounded above by the product of its diagonal entries:

$$0 \leq \det(A) \leq \prod_{i=1}^{n} A_{ii}.$$

This well-known result is sometimes called Hadamard's inequality [5, Theorem 7.8.1]. A lower bound of this form, however, is not possible without additional assumptions. Surprisingly, there is such an inequality when J is diagonally dominant with positive entries.

Theorem 1.1. *Let $n \geq 3$, and let J be an $n \times n$ symmetric diagonally dominant matrix with off-diagonal entries $m \geq J_{ij} \geq \ell > 0$. Then, the following inequality holds:*

$$\frac{\det(J)}{\prod_{i=1}^{n} J_{ii}} \geq \left(1 - \frac{1}{2(n-2)}\sqrt{\frac{m}{\ell}}\left(1 + \frac{m}{\ell}\right)\right)^{n-1}$$
$$\to \exp\left(-\frac{1}{2}\sqrt{\frac{m}{\ell}}\left(1 + \frac{m}{\ell}\right)\right) \quad \text{as } n \to \infty.$$

The result above was discovered in an attempt to prove the following difficult norm inequality [4]. Let $S = (n-2)I_n + \mathbf{1}_n\mathbf{1}_n^\top$ be the diagonally balanced matrix whose off-diagonal entries are all equal to 1 (I_n is the $n \times n$ identity matrix and $\mathbf{1}_n$ is the n-dimensional column vector consisting of all ones).

Theorem 1.2. *(See [4].) Let $n \geq 3$. For any symmetric diagonally dominant matrix J with $J_{ij} \geq \ell > 0$, we have*

$$\|J^{-1}\|_\infty \leq \frac{1}{\ell}\|S^{-1}\|_\infty = \frac{3n-4}{2\ell(n-2)(n-1)}.$$

Moreover, equality is achieved if and only if $J = \ell S$.

Here, $\|\cdot\|_\infty$ is the maximum absolute row sum of a matrix, which is the matrix norm induced by the infinity norm $|\cdot|_\infty$ on vectors in $\mathbb{R}^n$.

The bound in Theorem 1.1 depends on the largest off-diagonal entry of J (in an essential way; see Example 3.3), and thus is ill-adapted to prove Theorem 1.2. For instance, combining Theorem 1.1 with Hadamard's inequality applied to the positive definite $J^\star := J^{-1}\det(J)$ (the *adjugate* of J) in the obvious way gives estimates which are worse than Theorem 1.2. Nevertheless, Theorem 1.1 should be of independent interest, and we prove it in Section 2 using a block matrix factorization.

2. Proof of Theorem 1.1

Our arguments for proving Theorem 1.1 are inspired by block LU factorization ideas in [2]. For $1 \le i \le n$, let $J_{(i)}$ be the lower right $(n-i+1)\times(n-i+1)$ block of J, so $J_{(1)} = J$ and $J_{(n)} = (J_{nn})$. Also, for $1 \le i \le n-1$, let $b_{(i)} \in \mathbb{R}^{n-i}$ be the column vector such that

$$J_{(i)} = \begin{pmatrix} J_{ii} & b_{(i)}^\top \\ b_{(i)} & J_{(i+1)} \end{pmatrix}.$$

Then our block decomposition takes the form, for $1 \le i \le n-1$,

$$J_{(i)} = \begin{pmatrix} 1 & U_{(i)} \\ 0 & I_{n-i} \end{pmatrix} \begin{pmatrix} s_i & 0 \\ b_{(i)} & J_{(i+1)} \end{pmatrix}$$

with

$$s_i = J_{ii}\left(1 - \frac{b_{(i)}^\top J_{(i+1)}^{-1} b_{(i)}}{J_{ii}}\right) \quad \text{and} \quad U_{(i)} = b_{(i)}^\top J_{(i+1)}^{-1}.$$

Notice that $\det(J) = J_{nn}\prod_{i=1}^{n-1} s_i$, or equivalently,

$$\frac{\det(J)}{\prod_{i=1}^n J_{ii}} = \prod_{i=1}^{n-1} \frac{s_i}{J_{ii}} = \prod_{i=1}^{n-1}\left(1 - \frac{b_{(i)}^\top J_{(i+1)}^{-1} b_{(i)}}{J_{ii}}\right). \tag{1}$$

It remains to bound each factor s_i/J_{ii}. We first establish the following results.

Recall the *Loewner partial ordering* on symmetric matrices: $A \succeq B$ means that $A - B$ is positive semidefinite.

Lemma 2.1. *Let J be a symmetric diagonally balanced $n\times n$ matrix with $0 < \ell \le J_{ij} \le m$ for $i \ne j$. Then $\ell S \preceq J \preceq mS$, and the eigenvalues $\lambda_1 \le \cdots \le \lambda_n$ of J satisfy*

$$(n-2)\ell \le \lambda_i \le (n-2)m \quad \textit{for } 1 \le i \le n-1 \quad \textit{and} \quad 2(n-1)\ell \le \lambda_n \le 2(n-1)m.$$

Moreover, if J is diagonally dominant, then the lower bounds still hold.

Proof. We first show that if $P \geq 0$ is a symmetric diagonally dominant matrix, then $P \succeq 0$. For any $x \in \mathbb{R}^n$,

$$x^\top Px = \sum_{i=1}^{n} P_{ii}x_i^2 + 2\sum_{i<j} P_{ij}x_ix_j \geq \sum_{i=1}^{n}\left(\sum_{j\neq i} P_{ij}\right)x_i^2 + 2\sum_{i<j} P_{ij}x_ix_j$$
$$= \sum_{i<j} P_{ij}(x_i + x_j)^2 \geq 0.$$

Since the matrices $P = J - \ell S$ and $Q = mS - J$ are symmetric and diagonally balanced with nonnegative entries, it follows that $P, Q \succeq 0$ by the discussion above, which means $\ell S \preceq J \preceq mS$. The eigenvalues of S are $\{n-2, \dots, n-2, 2(n-1)\}$, so the result follows by an application of [5, Corollary 7.7.4]. If J is diagonally dominant, then $\ell S \preceq J$, and hence the lower bounds still hold. □

Lemma 2.2. *Let J be a symmetric diagonally balanced $n \times n$ matrix with $0 < \ell \leq J_{ij} \leq m$ for $i \neq j$. For each $1 \leq i \leq n$, let $J_{(i)}$ be the lower right $(n-i+1)\times(n-i+1)$ block of J as defined above, and suppose the eigenvalues of $J_{(i)}$ are $\lambda_1 \leq \cdots \leq \lambda_{n-i+1}$. Then*

$$(n-2)\ell \leq \lambda_j \leq (n-2)m \quad \textit{for } 1 \leq j \leq n-i \quad \textit{and}$$
$$(2n-i-1)\ell \leq \lambda_{n-i+1} \leq (2n-i-1)m.$$

Moreover, if J is diagonally dominant, then the lower bounds still hold.

Proof. Write $J_{(i)} = H + D$, where H is the $(n-i+1)\times(n-i+1)$ diagonally balanced matrix and D is diagonal with nonnegative entries. Note that $(i-1)\ell I \preceq D \preceq (i-1)mI$, so $(i-1)\ell I + H \preceq J_{(i)} \preceq (i-1)mI + H$. Thus by [5, Corollary 7.7.4] and by applying Lemma 2.1 to H, we get, for $1 \leq j \leq n-i$,

$$(n-2)\ell = (n-i-1)\ell + (i-1)\ell \leq \lambda_j \leq (n-i-1)m + (i-1)m = (n-2)m,$$

and for $j = n-i+1$,

$$(2n-i-1)\ell = 2(n-i)\ell + (i-1)\ell \leq \lambda_{n-i+1} \leq 2(n-i)m + (i-1)m = (2n-i-1)m.$$

If J is diagonally dominant, then $(i-1)\ell I + H \preceq J_{(i)}$ and hence the lower bounds still hold. □

Proof of Theorem 1.1. Suppose J is diagonally dominant. For each $1 \leq i \leq n-1$ we have $J_{ii} \geq \sum_{j\neq i} J_{ij} \geq b_{(i)}^\top \mathbf{1}_{n-i}$, and by Lemma 2.2, the maximum eigenvalue of $J_{(i+1)}^{-1}$ is at most $\frac{1}{(n-2)\ell}$. Thus,

$$\frac{b_{(i)}^\top J_{(i+1)}^{-1} b_{(i)}}{J_{ii}} \leq \frac{1}{(n-2)\ell}\frac{b_{(i)}^\top b_{(i)}}{J_{ii}} \leq \frac{1}{(n-2)\ell}\frac{b_{(i)}^\top b_{(i)}}{b_{(i)}^\top \mathbf{1}} \leq \frac{\sqrt{(n-i+1)}m}{(n-2)\ell}\frac{\sqrt{b_{(i)}^\top b_{(i)}}}{b_{(i)}^\top \mathbf{1}}.$$

Since each entry of $b_{(i)}$ is bounded by ℓ and m, the reverse Cauchy–Schwarz inequality [7, Chapter 5] gives us

$$\frac{b_{(i)}^\top J_{(i+1)}^{-1} b_{(i)}}{J_{ii}} \le \frac{\sqrt{(n-i+1)}m}{(n-2)\ell} \frac{\ell+m}{2\sqrt{\ell m(n-i+1)}} = \frac{1}{2(n-2)}\sqrt{\frac{m}{\ell}}\left(1+\frac{m}{\ell}\right).$$

Substituting this inequality into (1) gives us the desired bound. □

3. Examples

We close with several examples.

Example 3.1. The matrix $S = (n-2)I_n + \mathbf{1}_n\mathbf{1}_n^\top$ has eigenvalues $\{n-2, \dots, n-2, 2(n-1)\}$, so

$$\frac{\det(S)}{\prod_{i=1}^n S_{ii}} = \frac{2(n-2)^{n-1}(n-1)}{(n-1)^n} = 2\left(1-\frac{1}{n-1}\right)^{n-1} \to \frac{2}{e} \quad \text{as } n \to \infty. \qquad \square$$

Example 3.2. When J is strictly diagonally dominant, the ratio $\det(J)/\prod_{i=1}^n J_{ii}$ can be arbitrarily close to 1. For instance, consider $J = \alpha I_n + \mathbf{1}_n\mathbf{1}_n^\top$ with $\alpha \ge n-2$, which has eigenvalues $\{(n+\alpha), \alpha, \dots, \alpha\}$ so

$$\frac{\det(J)}{\prod_{i=1}^n J_{ii}} = \frac{(n+\alpha)\alpha^{n-1}}{(\alpha+1)^n} \to 1 \quad \text{as } \alpha \to \infty. \qquad \square$$

Example 3.3. The following example demonstrates that we need an upper bound on the entries of J in Theorem 1.1(a). Let $n = 2k$ for some $k \in \mathbb{N}$, and consider the matrix J in the following block form:

$$J = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \quad A = (km + k\ell - 2\ell)I_k + \ell\mathbf{1}_k\mathbf{1}_k^\top, \; B = m\mathbf{1}_k\mathbf{1}_k^\top.$$

By the determinant block formula (since A and B commute), we have

$$\begin{aligned} \det(J) &= \det\bigl(A^2 - B^2\bigr) \\ &= \det\bigl[(km + k\ell - 2\ell)^2 I_k + \bigl(2k\ell m + 3k\ell^2 - 4\ell^2 - km^2\bigr)\mathbf{1}_k\mathbf{1}_k^\top\bigr] \\ &= 4\ell(k-1)(km + k\ell - \ell) \cdot (km + k\ell - 2\ell)^{2k-2}, \end{aligned}$$

where the last equality is obtained by considering the eigenvalues of $A^2 - B^2$. Then

$$\begin{aligned} \frac{\det(J)}{\prod_{i=1}^n J_{ii}} &= \frac{4\ell(k-1)(km + k\ell - \ell) \cdot (km + k\ell - 2\ell)^{2k-2}}{(km + k\ell - \ell)^{2k}} \\ &\to \frac{4\ell}{\ell+m}\exp\left(-\frac{2\ell}{\ell+m}\right) \quad \text{as } k \to \infty. \end{aligned}$$

Note that the last quantity above tends to 0 as $m/\ell \to \infty$. □

Upon submission of this paper, we also conjectured the following. We thank Minghua Lin for allowing us to include his proof [6] of this conjecture.

Conjecture 3.4. *For a positive, diagonally balanced symmetric J, we have the bound:*

$$\frac{\det(J)}{\prod_{i=1}^{n} J_{ii}} \leq \frac{\det(S)}{(n-1)^n} = 2\left(1 - \frac{1}{n-1}\right)^{n-1} \to \frac{2}{e}.$$

Without loss of generality, we may assume $J_{ii} = 1$ for all i. Then we can write $J = I_n + B$, where B is a symmetric stochastic matrix with $B_{ii} = 0$ for all i. Recall that a (row) stochastic matrix is a square matrix of nonnegative real numbers with each row summing to 1.

Theorem 3.5 *(Minghua Lin). Let B be an $n \times n$ symmetric stochastic matrix with $B_{ii} = 0$ for all i. Then*

$$\det(I_n + B) \leq 2\left(1 - \frac{1}{n-1}\right)^{n-1}. \tag{2}$$

Moreover, this inequality is sharp.

We start with some lemmas that are needed in the proof.

Lemma 3.6. *If B is an $n \times n$ symmetric stochastic matrix with $B_{ii} = 0$ for all i, then $\operatorname{tr} B^2 \geq \frac{n}{n-1}$. Equality holds if and only if $B_{ij} = \frac{1}{n-1}$ for all $i \neq j$.*

Proof. By the Cauchy–Schwarz inequality,

$$\left(n^2 - n\right) \sum_{i \neq j} B_{ij}^2 \geq \left(\sum_{i \neq j} B_{ij}\right)^2 = n^2,$$

so

$$\operatorname{tr} B^2 = \sum_{i \neq j} B_{ij}^2 \geq \frac{n}{n-1}.$$

The equality case is trivial. □

Lemma 3.7. *For $a > 0$, the function $f(t) = (1 + at)(1 - t/a)^{a^2}$, $0 \leq t \leq a$, is decreasing.*

Proof. It suffices to show that $\widetilde{f}(t) = \log f(t)$ is decreasing for $0 < t < a$. Observing that

$$\widetilde{f}'(t) = \frac{a}{1 + at} - \frac{a}{1 - t/a} = -\frac{a(1 + a^2)t}{(1 + at)(a - t)} < 0,$$

the conclusion follows. □

The key to the proof of Theorem 3.5 is the following lemma.

Lemma 3.8. *(See [1] or [3, Eq. (1.2)].) Let A be an $n \times n$ positive semidefinite matrix. If $m = \frac{\operatorname{tr} A}{n}$ and $s = \sqrt{\frac{\operatorname{tr} A^2}{n} - m^2}$, then*

$$(m - s\sqrt{n-1})(m + s/\sqrt{n-1})^{n-1} \le \det A \le (m + s\sqrt{n-1})(m - s/\sqrt{n-1})^{n-1}.$$

Proof of Theorem 3.5. Let $A = I_n + B$ so that A is positive semidefinite. A calculation gives $m = \frac{\operatorname{tr} A}{n} = 1$ and $s^2 = \frac{\operatorname{tr} A^2}{n} - m^2 = \frac{\operatorname{tr} B^2}{n}$. Thus, by Lemma 3.8, we have

$$\det(I_n + B) \le (1 + s\sqrt{n-1})(1 - s/\sqrt{n-1})^{n-1}, \tag{3}$$

where $s = \sqrt{\frac{\operatorname{tr} B^2}{n}}$. Note that $\operatorname{tr} B^2 = \sum_{i \ne j} B_{ij}^2 < n^2 - n$ for $n \ge 3$, so $s < \sqrt{n-1}$. On the other hand, by Lemma 3.6, we have $\frac{\operatorname{tr} B^2}{n} \ge \frac{1}{n-1}$, so $s \ge \frac{1}{\sqrt{n-1}}$. By Lemma 3.7, we know $f(s) = (1 + s\sqrt{n-1})(1 - s/\sqrt{n-1})^{n-1}$ is decreasing with respect to $s \in [\frac{1}{\sqrt{n-1}}, \sqrt{n-1})$. Thus,

$$f(s) \le f\left(\frac{1}{\sqrt{n-1}}\right) = 2\left(1 - \frac{1}{n-1}\right)^{n-1}. \tag{4}$$

Inequality (2) now follows from (3) and (4).

Taking $B_{ij} = \frac{1}{n-1}$ for all $i \ne j$, equality in (2) holds. This proves the sharpness of (2). □

Remark 3.9. The lower bound of $\det A$ in (3) does not give a useful lower bound for $\det(I_n + B)$ in Theorem 3.5. Indeed, define $g(s) = (1 - s\sqrt{n-1})(1 + s/\sqrt{n-1})^{n-1}$ for $s = \sqrt{\frac{\operatorname{tr} B^2}{n}} \ge \frac{1}{\sqrt{n-1}}$. Then in order that $g(s) \ge 0$, we must have $s \le \frac{1}{\sqrt{n-1}}$, but $g(\frac{1}{\sqrt{n-1}}) = 0$.

Remark 3.10. In the proof of Theorem 3.5, we do not require that the entries of B be positive. Thus Theorem 3.5 is also valid for diagonally balanced symmetric matrices $I_n + B$ with entries of B negative.

References

[1] J.M. Borwein, G.P.H. Styan, H. Wolkowicz, Some inequalities involving statistical expressions: solution to problem 81-10 of L.V. Foster, SIAM Rev. 24 (1982) 340–342.
[2] J.W. Demmel, N.J. Higham, R. Schreiber, Block LU factorization, Research Institute of Advanced Computer Science, 1992.
[3] B. Grone, C. Johnson, E.M. De Sa, H. Wolkowicz, Improving Hadamard's inequality, Linear Multilinear Algebra 16 (1984) 305–322.
[4] C. Hillar, S. Lin, A. Wibisono, Tight bounds on the infinity norm of inverses of symmetric diagonally dominant positive matrices, arXiv:1301.3321, 2014.
[5] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, 1990.
[6] M. Lin, private communication, 2013.
[7] J.M. Steele, The Cauchy–Schwarz Master Class, Cambridge University Press, 2004.