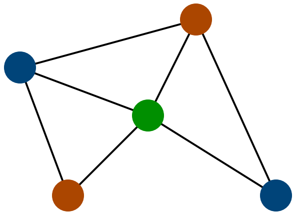


$$\Longrightarrow \qquad I_{G,k} = \langle g_1, \ldots, g_n \rangle$$

Algebraic Characterization of Uniquely Colorable Graphs

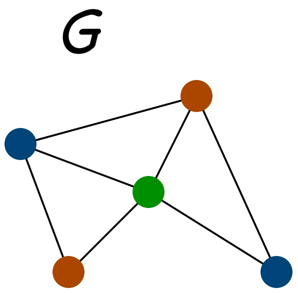
Christopher Hillar
(Texas A&M University)

Joint with Troels Windfeldt
(University of Copenhagen)

Outline of talk

- Introduction: Colorability
- Motivational Problem
- Previous Work: k-colorability
  - algebraic characterizations
  - Groebner bases
- Notation and Color Encodings
- Statement of Main Results
- Algorithms

Graph Colorings



Let G be a simple graph $G = (V,E)$ with vertices $V = \{1,2,\ldots,n\}$, edges E

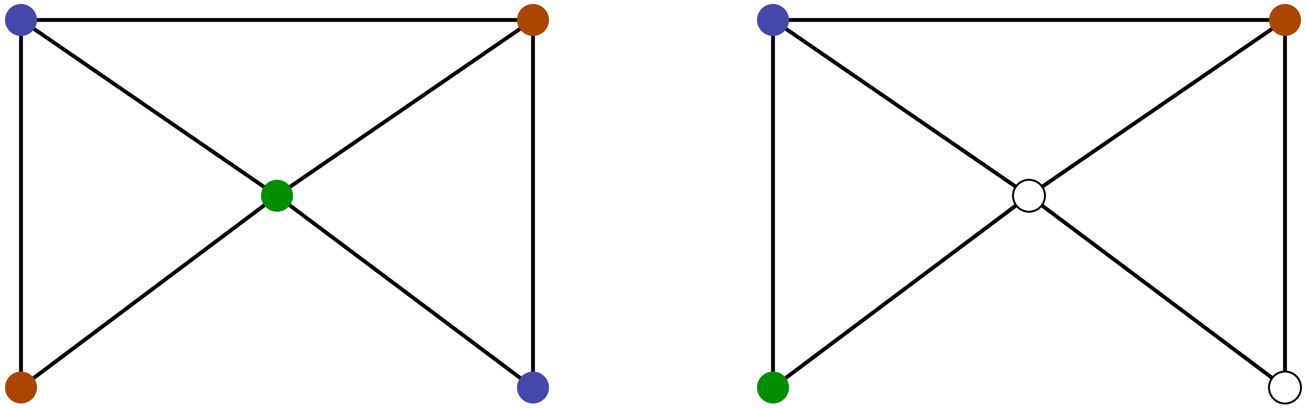
Def: A *k*-coloring of G is an assignment of k colors to the vertices of G

Def: A k-coloring is proper if adjacent vertices receive different colors

Def: A graph is *k*-colorable if it has a proper k-coloring

Unique Colorability

A 3-colorable graph that is not 2-colorable:



Other than permuting the colors, no other proper 3-colorings

Def: A **uniquely k-colorable** graph is one which has exactly $k!$ proper k-colorings. In this case, the coloring is unique up to permutation of colors

Technical Observation

G cannot be uniquely k-colorable if there is a proper k-coloring not using all k colors

Not uniquely 3-colorable although there are 3! = 6 proper 3-colorings of G that use all 3 colors

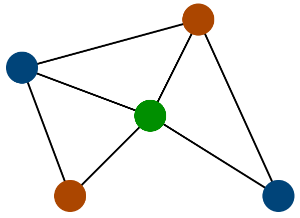
Motivational Problem

In 1990, Xu showed that uniquely k-colorable graphs must have many edges

Theorem (Xu): If G is uniquely k-colorable then

$$|E| \geq (k-1)n - k(k-1)/2$$

In example, $7 \geq 2 \cdot 5 - 3 = 7$



Notice: G contains a triangle

Xu's Conjecture

In general, he conjectured that if equality holds, G must contain a k-clique.

Conjecture (Xu): If G is uniquely k-colorable and $|E| = (k-1)n - k(k-1)/2$, then G contains a k-clique

This conjecture was shown to be false in 2001 by Akbari et al using a technical combinatorial argument

We wanted to find a JPE proof (Just Press Enter)

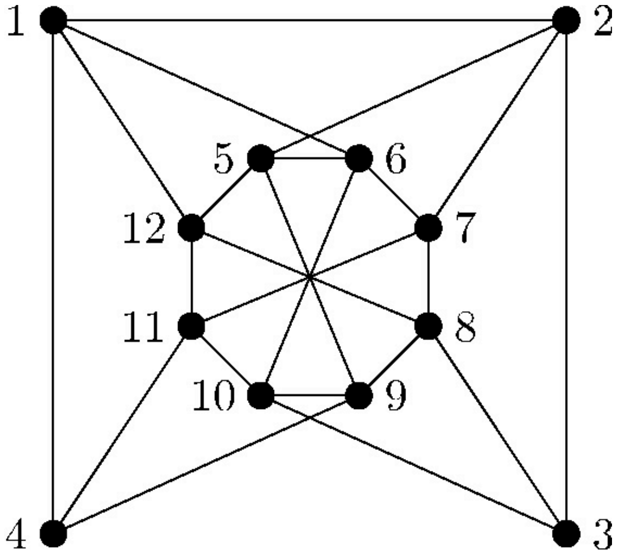
Computer Proof

This leads to the following concrete problem:

Problem: Find an (effective) algorithm to decide unique *k*-colorability (and find the coloring).

3-colorable?

Uniquely 3-colorable?



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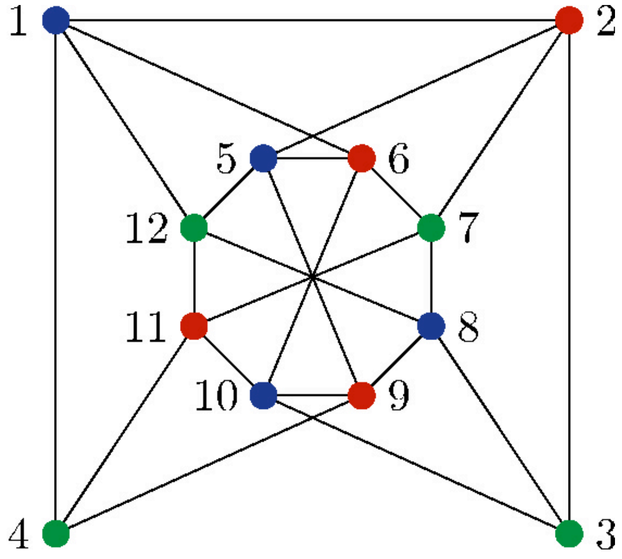

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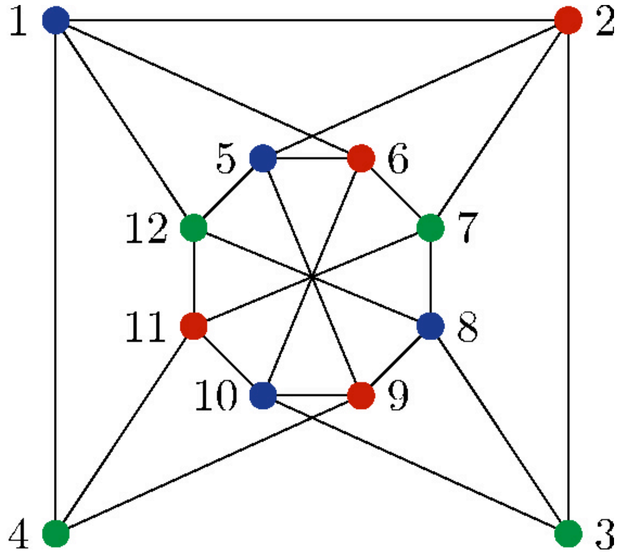

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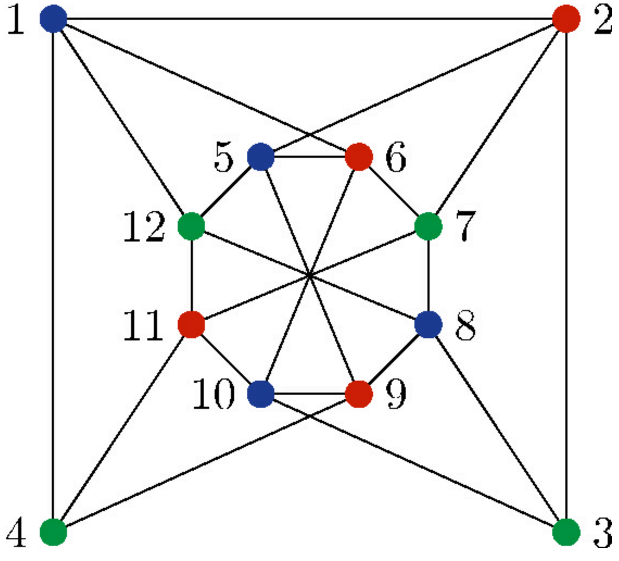

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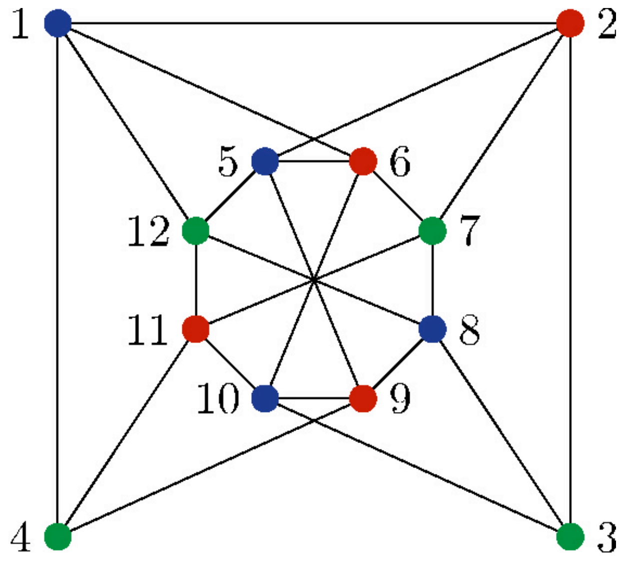
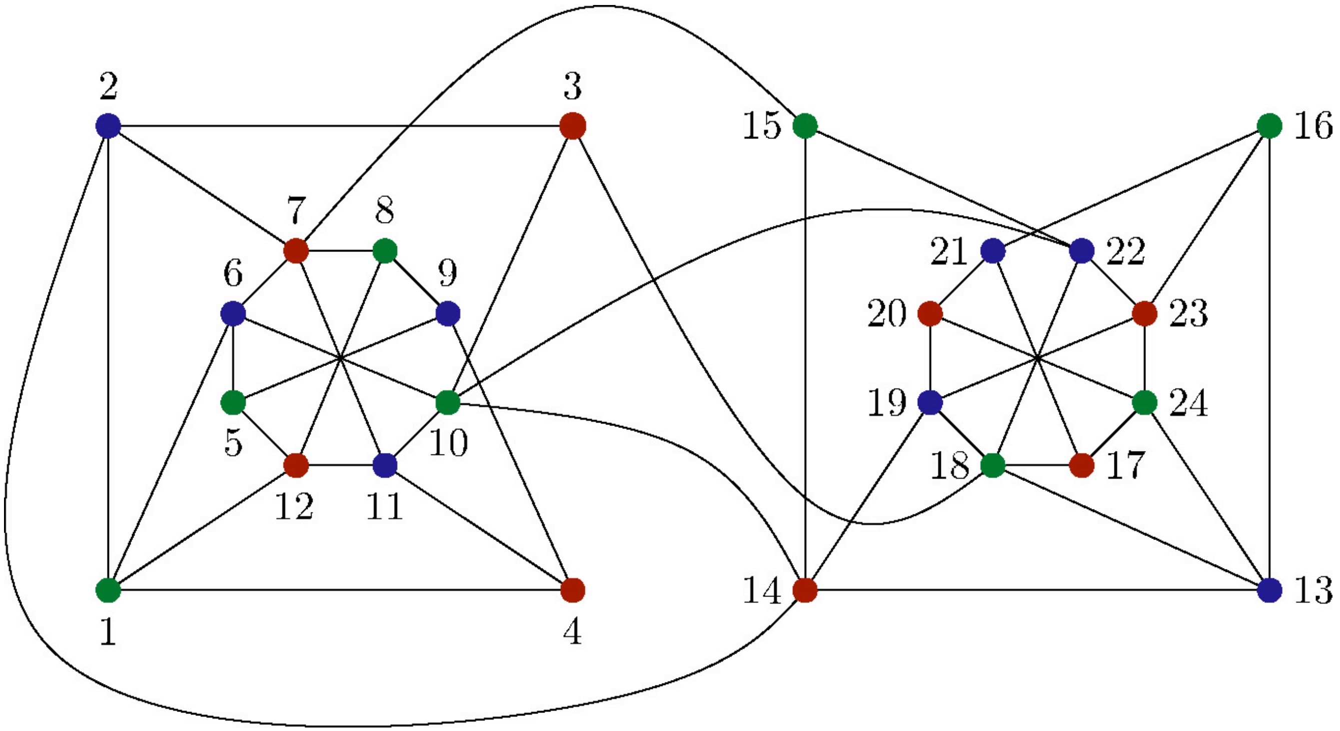


Counterexample to Xu

How about the graph actually used
by Akbari, Mirrokni, Sadjad: $|E| = 45$, $n = 24$, $k = 3$



Algebraic Colorability

Main Idea (implicit in work of Bayer, de Loera, Lovász):

colorings are points in varieties

⇓

varieties are represented by ideals

⇓

ideals can be manipulated with **Groebner Bases**

k-Colorings as Points in Varieties

Setup: F is an algebraically closed field, (char F) $\nmid k$

So F contains k distinct kth roots of unity. Let

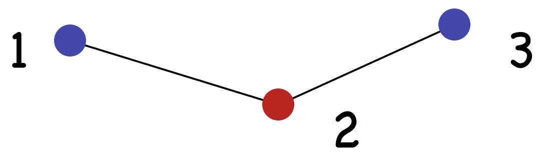
$$I_k = \langle x_1^k - 1,\ x_2^k - 1,\ \ldots,\ x_n^k - 1 \rangle \subset F[x_1,\ldots,x_n]$$

This ideal is radical, and $|V(I_k)| = k^n$

Can think of point $v = (v_1,\ldots,v_n)$ in $V(I_k)$ as assignment

$$v = (v_1,\ldots,v_n) \iff \text{vertex } i \text{ gets color } v_i$$

Eg. If 1 = Blue, -1 = Red, then $v = (1,-1,1)$ is coloring



Proper k-Colorings of Graphs

We can also restrict to proper k-colorings of graph G

$$I_{G,k} = I_k + \langle x_i^{k-1} + x_i^{k-2}x_j + \cdots + x_j^{k-1} : (i,j) \in E \rangle$$

This ideal is radical, and $|V(I_{G,k})|$ = # proper k-colorings

---

Proof: (=>) If v in $V(I_{G,k})$, wts v proper. If $v_i = v_j$ for $(i,j) \in E$, then

$$0 = v_i^{k-1} + v_i^{k-2}v_j + \cdots + v_j^{k-1} = kv_i^{k-1}$$ 🚫

(<=) If v proper, then $v_i \neq v_j$ and

$$(v_i - v_j) \cdot (v_i^{k-1} + \cdots + v_j^{k-1}) = v_i^k - v_j^k = 1 - 1 = 0$$

Thus, $v_i^{k-1} + \cdots + v_j^{k-1} = 0$ and v in $V(I_{G,k})$

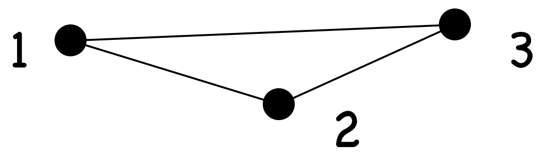
Algebraic Characterization

Notice that if $I_{G,k} = \langle 1 \rangle = F[x_1,\ldots,x_n]$ then

$$V(I_{G,k}) = \varnothing \quad \Rightarrow \quad G \text{ is not } k\text{-colorable}$$

Therefore, we have a test for *k*-colorability:

Algorithm: Compute a reduced Groebner basis B for $I_{G,k}$. Then, $B = \{1\}$ iff G is not k-colorable.



($k = 2$)

$$\begin{aligned} I_{G,k} &= \langle x_1^2 - 1,\ x_2^2 - 1,\ x_3^2 - 1, \\ &\qquad x_1 + x_2,\ x_2 + x_3,\ x_1 + x_3 \rangle \\ &= \langle 1 \rangle \end{aligned}$$

$$2x_1^2 = (x_1 - x_2)(x_1 + x_2) + (x_2 - x_3)(x_2 + x_3) + (x_1 - x_3)(x_1 + x_3)$$

Graph Polynomial

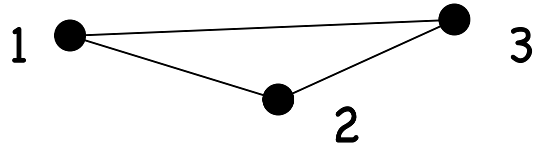
The graph polynomial of G encodes adjacency of vertices algebraically:

$$f_G = \prod_{\substack{\{i,j\}\in E \\ i<j}} (x_i - x_j)$$

One should think of f_G as a test polynomial for k-colorability.

$I_3 = \langle x_1^3 - 1,\ x_2^3 - 1,\ x_3^3 - 1 \rangle$

$k = 3$, $n = 3$



Notice that $f_G = (x_1 - x_2)(x_2 - x_3)(x_1 - x_3) \notin I_k$

Conclusion: There is a 3-coloring that is proper!

Characterization Theorem

k-colorability Theorem (Bayer, de Loera, Alon, Tarsi, Mnuk, Kleitman, Lovász): The following are equivalent

(1) G is not k-colorable
(2) $I_{G,k} = \langle 1 \rangle$
(3) f_G is contained in I_k (colorings zero f_G)

Corollary: There are simple tests for k-colorability involving polynomial algebra.

Our goal: Develop a similar characterization theorem for unique k-colorability and give a complete description of $I_{G,k}$ when G is k-colorable

Preparation: Color classes

Given G which is k-colorable with a coloring $v = (v_1,\dots,v_n)$ using all k colors, we define:

Color class of i = $\mathrm{cl}(i) = \{ j : v_j = v_i \}$

representative of $\mathrm{cl}(i)$ = $\max\{ j : j \text{ is in } \mathrm{cl}(i) \}$

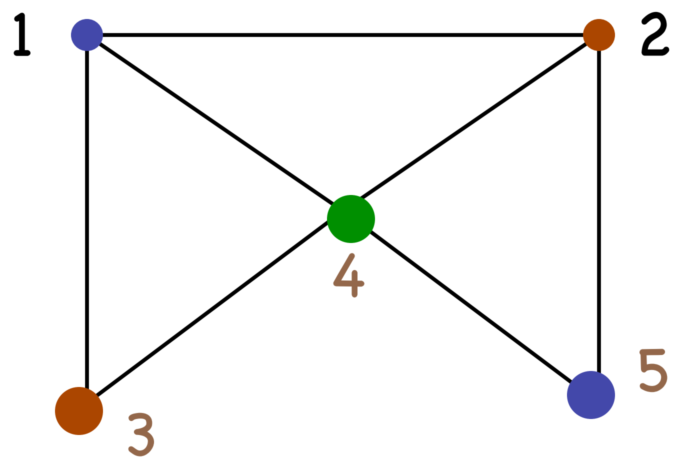
Denote these representatives

$$m_1 < m_2 < \dots < m_k = n$$

Eg. $\mathrm{cl}(1) = \{1,5\}$, $\mathrm{cl}(2) = \{2,3\}$

$\mathrm{cl}(4) = \{4\}$

$$m_1 = 3 < m_2 = 4 < m_3 = 5$$



New Polynomial Encoding

We need a replacement for the graph polynomial f_G in the statement of the k-colorability theorem

Def: Let $U \subseteq \{1,\ldots,n\}$. Then we set h_U^d to be the sum of all monomials of degree d in the variables $\{x_i : i \text{ in } U\}$

Eg. $U = \{1,2,3\}$, $d = 2$

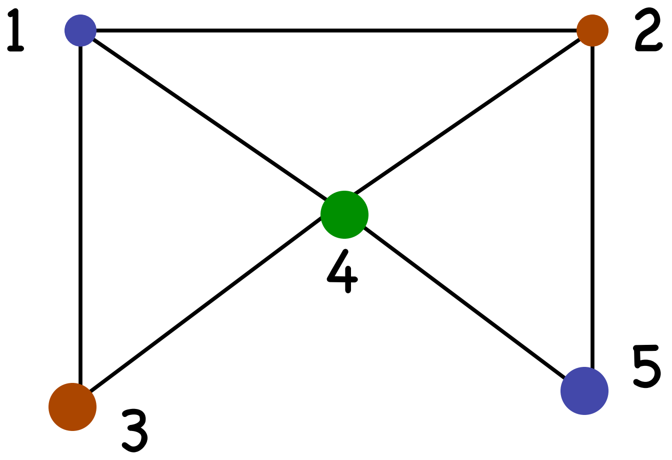
$$h_U^d = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$$

Also, we set $h_U^0 = 1$

Replacements for f_G

Def: Given a proper k-coloring, for each vertex i, let

$$g_i = \begin{cases} x_i^k - 1 & i = m_k \ (= n) \\ h^j_{\{m_j,\dots,m_k\}} & i = m_j \text{ for some } j \neq k \\ h^1_{\{i,m_2\dots,m_k\}} & i \text{ in } \mathrm{cl}(m_1) \\ x_i - x_{\max\ \mathrm{cl}(i)} & \text{otherwise} \end{cases}$$



$g_5 = x_5^3 - 1$

$g_4 = h^2_{\{4,5\}} = x_4^2 + x_4x_5 + x_5^2$

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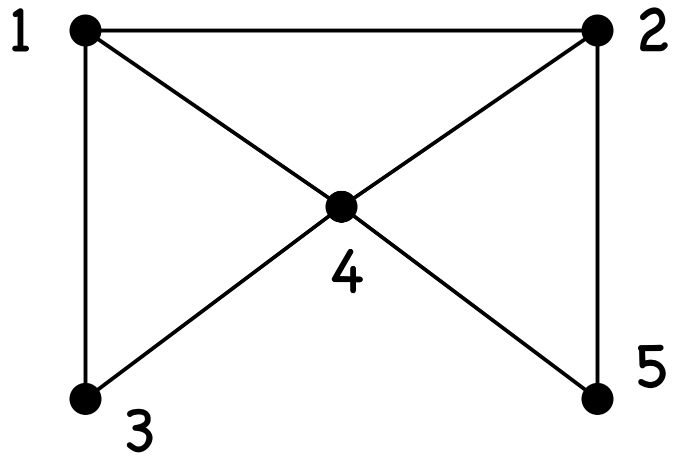
$g_1 = x_1 - x_5$

$(m_1,m_2,m_3) = (3,4,5)$

Can go backwards

Find a proper k-coloring giving a set

$$g_i = \begin{cases} x_i^k - 1 & i = m_k \ (= n) \\ h^j_{\{m_j,\ldots,m_k\}} & i = m_j \text{ for some } j \neq k \\ h^1_{\{i,m_2,\ldots,m_k\}} & i \text{ in } \mathrm{cl}(m_1) \\ x_i - x_{\max \mathrm{cl}(i)} & \text{otherwise} \end{cases}$$



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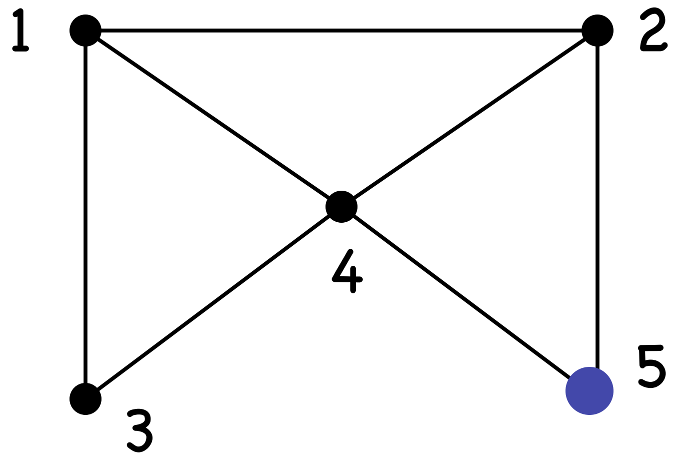

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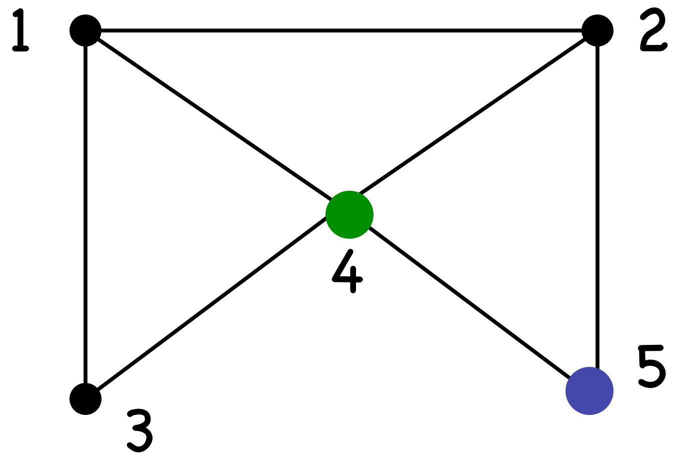

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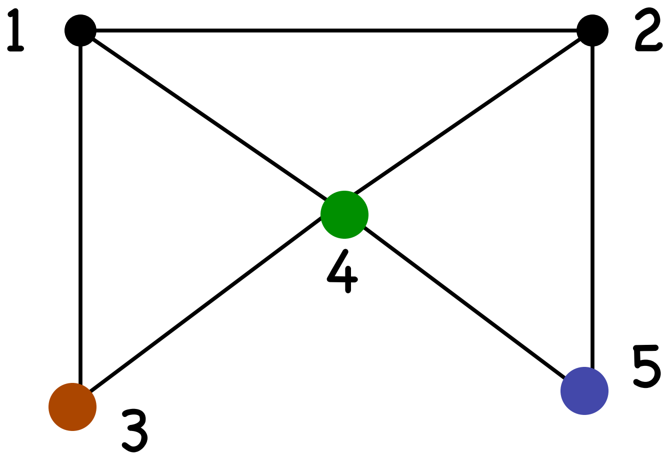

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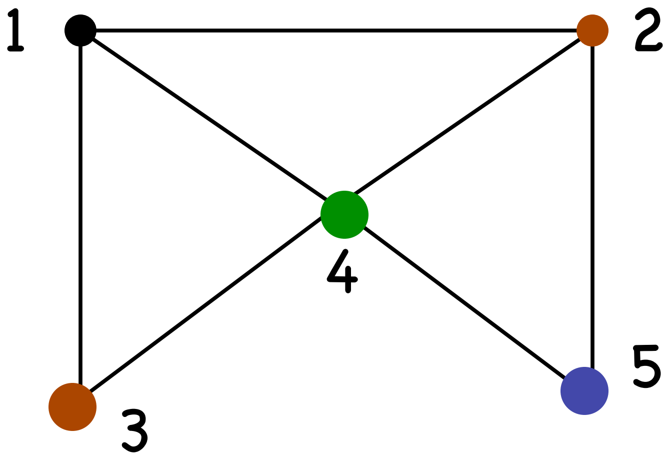

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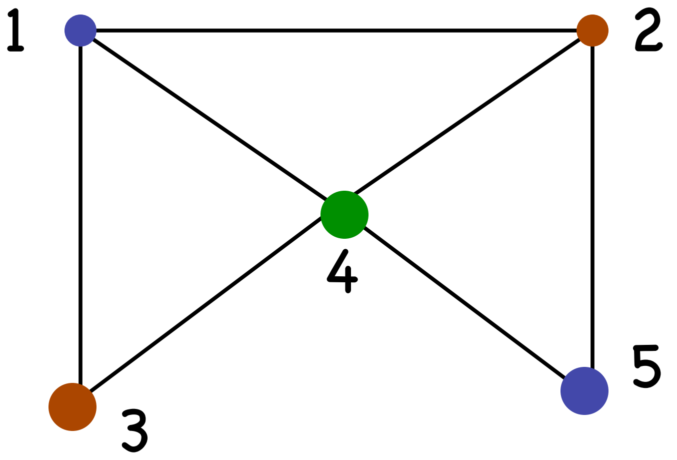

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Encoding: v --> $A_v = \{g_1,\ldots,g_n\}$

The set of polynomials $\{g_1,\ldots,g_n\}$ encodes the coloring v

Lemma: Let $A_v = \langle g_1,\ldots,g_n \rangle$

(1) $I_{G,k} \subseteq A_v$

(2) A_v is radical

(3) $|V(A_v)| = k!$

Interpretation:

(1) zeroes of A_v are proper k-colorings of G

(2) A_v and its zeroes are in 1-1 correspondence

(3) Up to permutation, A_v encodes precisely v

Characterization Theorem

Theorem [-,W 06]: Let G be a graph. The following are equivalent

(1) G is k-colorable

(2) $\cap A_v \subseteq I_{G,k}$

(3) $\cap A_v = I_{G,k}$

Point: We have found an interpretation involving ideals for the statement:

$$\cup \{ V(A_v) : v \text{ is proper} \} = \{ \text{proper colorings} \}$$

Unique Characterization

In general, the map from proper k-colorings

$$v \dashrightarrow A_v = \{g_1,\ldots,g_n\}$$

only depends on how v partitions $V = \{1,\ldots,n\}$ into color classes. In particular,

Fact: If G is uniquely colorable, then there is a unique set of polynomials $\{g_1,\ldots,g_n\}$ that corresponds to all v.

Proof: All v partition V the same way

Unique Characterization

Corollary [-,W 06]: Fix a proper k-coloring v of G. Let $A_v = \langle g_1,\dots,g_n \rangle$. The following are equivalent

(1) G is uniquely k-colorable
(2) $g_1,\dots,g_n$ belong to $I_{G,k}$
(3) $A_v = \langle g_1,\dots,g_n \rangle = I_{G,k}$

More canonically, we have the following

Theorem [-,W 06]: G is uniquely k-colorable if and only if the reduced Groebner basis for $I_{G,k}$ (w.r.t any term order with $x_n < \cdots < x_1$) has the form $g_1,\dots,g_n$

Algorithms

The main theorems give algorithms for determining unique colorability

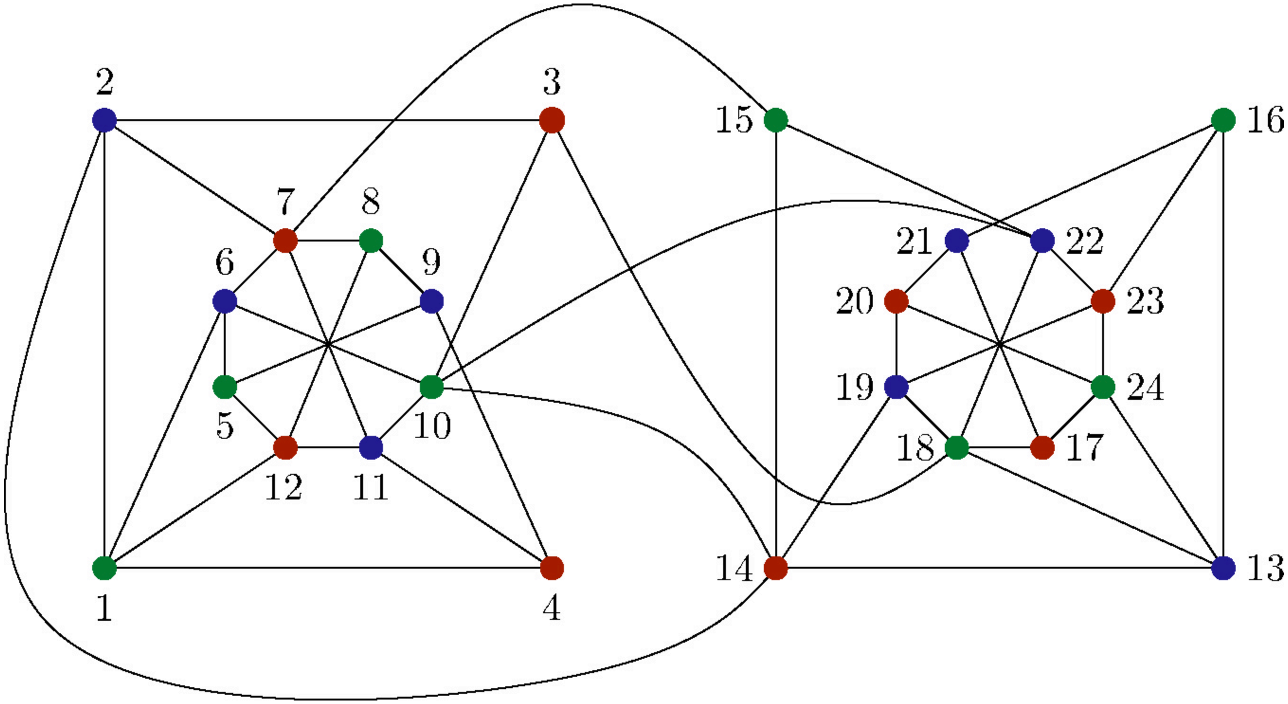
Algorithm 1: Given a proper k-coloring, construct the polynomials $g_1,\ldots,g_n$ and reduce them modulo $I_{G,k}$

Algorithm 2: Compute the reduced Groebner basis for $I_{G,k}$ and see whether it has the form $g_1,\ldots,g_n$; if so, read off the coloring.

- (easy to check form and to read off coloring)

Just Press Enter

Computing with a field F with char $F = 2$, we find that (after pressing enter and waiting 5 seconds) the following graph is indeed uniquely k-colorable.



The End

(of talk)