Algebraic Characterization of Uniquely Colorable Graphs

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Outline of talk

• Introduction: Colorability
• Motivational Problem
• Previous Work: \( k \)-colorability
  – algebraic characterizations
  – Groebner bases
• Notation and Color Encodings
• Statement of Main Results
• Algorithms
Graph Colorings

Let $G$ be a simple graph $G = (V,E)$ with vertices $V = \{1,2,...,n\}$, edges $E$

**Def:** A $k$-coloring of $G$ is an assignment of $k$ colors to the vertices of $G$

**Def:** A $k$-coloring is **proper** if adjacent vertices receive different colors

**Def:** A graph is $k$-colorable if it has a proper $k$-coloring
Unique Colorability

A 3-colorable graph that is not 2-colorable:

Other than permuting the colors, no other proper 3-colorings

**Def:** A *uniquely k-colorable* graph is one which has exactly $k!$ proper $k$-colorings. In this case, the coloring is unique up to permutation of colors.
Technical Observation

$G$ cannot be uniquely $k$-colorable if there is a proper $k$-coloring not using all $k$ colors

Not uniquely 3-colorable although there are $3! = 6$ proper 3-colorings of $G$ that use all 3 colors
Motivational Problem

In 1990, Xu showed that uniquely $k$-colorable graphs must have many edges

**Theorem** (Xu): If $G$ is uniquely $k$-colorable then

$$|E| \geq (k-1)n - k(k-1)/2$$

In example, $7 \geq 2 \cdot 5 - 3 = 7$

Notice: $G$ contains a triangle
Xu’s Conjecture

In general, he conjectured that if equality holds, $G$ must contain a $k$-clique.

**Conjecture (Xu):** If $G$ is uniquely $k$-colorable and $|E| = (k-1)n - k(k-1)/2$, then $G$ contains a $k$-clique.

This conjecture was shown to be false in 2001 by Akbari et al using a technical combinatorial argument.

We wanted to find a JPE proof (Just Press Enter)
Computer Proof

This leads to the following concrete problem:

**Problem:** Find an (effective) algorithm to decide unique $k$-colorability (and find the coloring).

3-colorable?

Uniquely 3-colorable?
Computer Proof

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Uniquely 3-colorable?  

**Yes!**
Computer Proof

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**Problem:** Find an (effective) algorithm to decide unique $k$-colorability (and find the coloring).

3-colorable?

Uniquely 3-colorable?

*Yes!*

*Yes!*
Computer Proof

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**Problem:** Find an (effective) algorithm to decide unique $k$-colorability (and find the coloring).

3-colorable?

Uniquely 3-colorable?

**Yes!**

**Yes!**

...
Counterexample to Xu

How about the graph actually used by Akbari, Mirrokni, Sadjad: $|E| = 45$, $n = 24$, $k = 3$
Algebraic Colorability

Main Idea (implicit in work of Bayer, de Loera, Lovász):

- colorings are points in varieties
  \[\downarrow\]
- varieties are represented by ideals
  \[\downarrow\]
- ideals can be manipulated with Groebner Bases
$k$-Colorings as Points in Varieties

**Setup:** $F$ is an algebraically closed field, $(\text{char } F) \nmid k$

So $F$ contains $k$ distinct $k$th roots of unity. Let

$$I_k = \langle x_1^k - 1, x_2^k - 1, \ldots, x_n^k - 1 \rangle \subset F[x_1, \ldots, x_n]$$

This ideal is radical, and $|V(I_k)| = k^n$

Can think of point $v = (v_1, \ldots, v_n)$ in $V(I_k)$ as assignment

$v = (v_1, \ldots, v_n) \iff \text{vertex } i \text{ gets color } v_i$

Eg. If $1 = \text{Blue}, -1 = \text{Red}$, then $v = (1, -1, 1)$ is coloring

![Diagram](attachment:diagram.png)
Proper $k$-Colorings of Graphs

We can also restrict to proper $k$-colorings of graph $G$

$$I_{G,k} = I_k + \langle x_i^{k-1} + x_i^{k-2}x_j + \cdots + x_j^{k-1} : (i,j) \in E \rangle$$

This ideal is radical, and $|V(I_{G,k})| = \#$ proper $k$-colorings

Proof: ($\Rightarrow$) If $v$ in $V(I_{G,k})$, wts $v$ proper. If $v_i = v_j$ for $(i,j) \in E$, then

$$0 = v_i^{k-1} + v_i^{k-2}v_j + \cdots + v_j^{k-1} = kv_i^{k-1}$$

($\Leftarrow$) If $v$ proper, then $v_i \neq v_j$ and

$$(v_i - v_j) \cdot (v_i^{k-1} + \cdots + v_j^{k-1}) = v_i^k - v_j^k = 1 - 1 = 0$$

Thus, $v_i^{k-1} + \cdots + v_j^{k-1} = 0$ and $v$ in $V(I_{G,k})$
Algebraic Characterization

Notice that if \( I_{G,k} = \langle 1 \rangle = F[x_1, \ldots, x_n] \) then

\[
V(I_{G,k}) = \emptyset \implies G \text{ is not } k\text{-colorable}
\]

Therefore, we have a test for \( k\)-colorability:

**Algorithm:** Compute a reduced Groebner basis \( B \) for \( I_{G,k} \). Then, \( B = \{1\} \) iff \( G \) is not \( k\)-colorable.

\[
I_{G,k} = \langle x_1^2 - 1, \ x_2^2 - 1, \ x_3^2 - 1, \ x_1 + x_2, \ x_2 + x_3, \ x_1 + x_3 \rangle = \langle 1 \rangle
\]

\[
2x_1^2 = (x_1-x_2)(x_1+x_2) + (x_2-x_3)(x_2+x_3) + (x_1-x_3)(x_1+x_3)
\]
Graph Polynomial

The graph polynomial of $G$ encodes adjacency of vertices algebraically:

$$f_G = \prod_{\{i,j\} \in E, i<j} (x_i - x_j)$$

One should think of $f_G$ as a test polynomial for $k$-colorability. $I_3 = \langle x_1^3 - 1, x_2^3 - 1, x_3^3 - 1 \rangle$

$k = 3, n = 3$

Notice that $f_G = (x_1-x_2)(x_2-x_3)(x_1-x_3) \notin I_k$

Conclusion: There is a 3-coloring that is proper!
Characterization Theorem

\textbf{\(k\)-colorability Theorem} (Bayer, de Loera, Alon, Tarsi, Mnuk, Kleitman, Lovász): The following are equivalent

1. \(G\) is not \(k\)-colorable
2. \(I_{G,k} = \langle 1 \rangle\)
3. \(f_G\) is contained in \(I_k\) (colorings zero \(f_G\))

\textbf{Corollary}: There are simple tests for \(k\)-colorability involving polynomial algebra.

\textbf{Our goal}: Develop a similar characterization theorem for unique \(k\)-colorability and give a complete description of \(I_{G,k}\) when \(G\) is \(k\)-colorable
Preparation: Color classes

Given $G$ which is $k$-colorable with a coloring $v = (v_1, \ldots, v_n)$ using all $k$ colors, we define:

**Color class of $i$** = $\text{cl}(i) = \{ j : v_j = v_i \}$

**representative of** $\text{cl}(i) = \max\{ j : j \text{ is in } \text{cl}(i) \}$

Denote these representatives

\[ m_1 < m_2 < \ldots < m_k = n \]

**Eg.** $\text{cl}(1) = \{1, 5\}$, $\text{cl}(2) = \{2, 3\}$

$\text{cl}(4) = \{4\}$

\[ m_1 = 3 < m_2 = 4 < m_3 = 5 \]
New Polynomial Encoding

We need a replacement for the graph polynomial $f_G$ in the statement of the $k$-colorability theorem.

**Def:** Let $U \subseteq \{1,\ldots,n\}$. Then we set $h_U^d$ to be the sum of all monomials of degree $d$ in the variables $\{x_i : i \in U\}$.

Eg. $U = \{1,2,3\}$, $d = 2$

$$h_U^d = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_1 x_3 + x_2 x_3$$

Also, we set $h_U^0 = 1$
Replacements for $f_G$

**Def:** Given a proper $k$-coloring, for each vertex $i$, let

$$g_i = \begin{cases} 
x_i^k - 1 \\
h_j^{\{m_j, \ldots, m_k\}} \\
h_1^{\{i, m_2, \ldots, m_k\}} \\
x_i - x_{\text{max cl}(i)} \
\end{cases}$$

- $i = m_k (= n)$
- $i = m_j$ for some $j \neq k$
- $i$ in $\text{cl}(m_1)$
- otherwise

$g_5 = x_5^3 - 1$
$g_4 = h_2^{\{4, 5\}} = x_4^2 + x_4 x_5 + x_5^2$
$g_3 = h_1^{\{3, 4, 5\}} = x_3 + x_4 + x_5$
$g_2 = h_1^{\{2, 4, 5\}} = x_2 + x_4 + x_5$
$g_1 = x_1 - x_5$

$(m_1, m_2, m_3) = (3, 4, 5)$
Can go backwards

Find a proper $k$-coloring giving a set

$$g_i = \begin{cases} 
  x_i^k - 1 & i = m_k (= n) \\
  h^j_{\{m_j, \ldots, m_k\}} & i = m_j \text{ for some } j \neq k \\
  h^1_{\{i, m_2, \ldots, m_k\}} & i \text{ in } \text{cl}(m_1) \\
  x_i - x_{\max \text{ cl}(i)} & \text{otherwise}
\end{cases}$$

$g_5 = x_5^3 - 1$
$g_4 = h^2_{\{4, 5\}} = x_4^2 + x_4x_5 + x_5^2$
$g_3 = h^1_{\{3, 4, 5\}} = x_3 + x_4 + x_5$
$g_2 = h^1_{\{2, 4, 5\}} = x_2 + x_4 + x_5$
$g_1 = x_1 - x_5$  \hspace{1cm} (m_1, m_2, m_3) = (?, ?, ?)
Can go backwards

Find a proper $k$-coloring giving a set $g_i = \begin{cases} x_i^k - 1 & i = m_k (= n) \\ h_j^{\{m_j,...,mk\}} & i = m_j \text{ for some } j \neq k \\ h_1^{\{i,m_2,...,mk\}} & i \text{ in } cl(m_1) \\ x_i - x_{\text{max } cl(i)} & \text{otherwise} \end{cases}$

$g_5 = x_5^3 - 1$
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$g_1 = x_1 - x_5 \quad (m_1,m_2,m_3) = (?,?,5)$
Can go backwards

Find a proper $k$-coloring giving a set

$$g_i = \begin{cases} 
  x_i^k - 1 & i = m_k (= n) \\
  h_j^{\{m_j, \ldots, m_k\}} & i = m_j \text{ for some } j \neq k \\
  h_i^{\{i, m_2, \ldots, m_k\}} & i \text{ in } \text{cl}(m_1) \\
  x_i - x_{\max \text{ cl}(i)} & \text{otherwise} 
\end{cases}$$

$$g_5 = x_5^3 - 1$$
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$$g_3 = h^{\{3,4,5\}} = x_3 + x_4 + x_5$$
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$$g_1 = x_1 - x_5$$

$(m_1, m_2, m_3) = (?, 4, 5)$
Can go backwards

Find a proper $k$-coloring giving a set

$$g_i = \begin{cases} 
  x_i^k - 1 & i = m_k (~= n) \\
  h^j_{\{m_j,\ldots,m_k\}} & i = m_j \text{ for some } j \neq k \\
  h^i_{\{i,m_2,\ldots,m_k\}} & i \text{ in } \text{cl}(m_1) \\
  x_i - x_{\max cl(i)} & \text{otherwise}
\end{cases}$$

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$g_1 = x_1 - x_5 \quad (m_1, m_2, m_3) = (3,4,5)$
Encoding: \( \nu \rightarrow A_{\nu} = \{g_1, \ldots, g_n\} \)

The set of polynomials \( \{g_1, \ldots, g_n\} \) encodes the coloring \( \nu \)

**Lemma:** Let \( A_{\nu} = \langle g_1, \ldots, g_n \rangle \)

(1) \( I_{G,k} \subseteq A_{\nu} \)

(2) \( A_{\nu} \) is radical

(3) \( |V(A_{\nu})| = k! \)

**Interpretation:**

(1) zeroes of \( A_{\nu} \) are proper \( k \)-colorings of \( G \)

(2) \( A_{\nu} \) and its zeroes are in 1-1 correspondence

(3) Up to permutation, \( A_{\nu} \) encodes precisely \( \nu \)
Characterization Theorem

**Theorem** [–,W 06]: Let $G$ be a graph. The following are equivalent

1. $G$ is $k$-colorable
2. $\bigcap A_v \subseteq I_{G,k}$
3. $\bigcap A_v = I_{G,k}$

**Point:** We have found an interpretation involving ideals for the statement:

$$\bigcup \{ V(A_v) : v \text{ is proper } \} = \{ \text{ proper colorings } \}$$
Unique Characterization

In general, the map from proper $k$-colorings

$$v \rightarrow A_v = \{g_1, \ldots, g_n\}$$

only depends on how $v$ partitions $V = \{1, \ldots, n\}$ into color classes. In particular,

Fact: If $G$ is uniquely colorable, then there is a unique set of polynomials $\{g_1, \ldots, g_n\}$ that corresponds to all $v$.

Proof: All $v$ partition $V$ the same way
Unique Characterization

**Corollary** [-,W 06]: Fix a proper $k$-coloring $v$ of $G$. Let $A_v = \langle g_1, \ldots, g_n \rangle$. The following are equivalent

1. $G$ is uniquely $k$-colorable
2. $g_1, \ldots, g_n$ belong to $I_{G,k}$
3. $A_v = \langle g_1, \ldots, g_n \rangle = I_{G,k}$

More canonically, we have the following

**Theorem** [-,W 06]: $G$ is uniquely $k$-colorable if and only if the reduced Groebner basis for $I_{G,k}$ (w.r.t any term order with $x_n < \cdots < x_1$) has the form $g_1, \ldots, g_n$
Algorithms

The main theorems give algorithms for determining unique colorability

Algorithm 1: Given a proper $k$-coloring, construct the polynomials $g_1,\ldots,g_n$ and reduce them modulo $I_{G,k}$

Algorithm 2: Compute the reduced Groebner basis for $I_{G,k}$ and see whether it has the form $g_1,\ldots,g_n$; if so, read off the coloring.

- (easy to check form and to read off coloring)
Computing with a field $F$ with $\text{char } F = 2$, we find that (after pressing enter and waiting 5 seconds) the following graph is indeed uniquely $k$-colorable.
The End
(of talk)