

MINIMAL GENERATORS FOR SYMMETRIC IDEALS

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ABSTRACT. Let $R = K[X]$ be the polynomial ring in infinitely many indeterminates X over a field K , and let \mathfrak{S}_X be the symmetric group of X . The group \mathfrak{S}_X acts naturally on R , and this in turn gives R the structure of a module over the group ring $R[\mathfrak{S}_X]$. A recent theorem of Aschenbrenner and Hillar states that the module R is Noetherian. We address whether submodules of R can have any number of minimal generators, answering this question positively.

Let $R = K[X]$ be the polynomial ring in infinitely many indeterminates X over a field K . Write \mathfrak{S}_X (resp. \mathfrak{S}_N) for the symmetric group of X (resp. $\{1, \dots, N\}$) and $R[\mathfrak{S}_X]$ for its (left) group ring, which acts naturally on R . A *symmetric ideal* $I \subseteq R$ is an $R[\mathfrak{S}_X]$ -submodule of R .

Aschenbrenner and Hillar recently proved [1] that all symmetric ideals are finitely generated over $R[\mathfrak{S}_X]$. They were motivated by finiteness questions in chemistry [2] and algebraic statistics [4]. In proving the Noetherianity of R , it was shown that a symmetric ideal I has a special, finite set of generators called a *minimal Gröbner basis*. However, the more basic question whether I is always cyclic (already asked by Josef Schicho [3]) was left unanswered in [1]. Our result addresses a generalization of this important issue.

Theorem 1. *For every positive integer n , there are symmetric ideals of R generated by n polynomials which cannot have fewer than n $R[\mathfrak{S}_X]$ -generators.*

In what follows, we work with the set $X = \{x_1, x_2, x_3, \dots\}$, although as remarked in [1], this is not really a restriction. In this case, \mathfrak{S}_X is naturally identified with \mathfrak{S}_∞ , the permutations of the positive integers, and $\sigma x_i = x_{\sigma i}$ for $\sigma \in \mathfrak{S}_\infty$.

Let M be a finite multiset of positive integers and let i_1, \dots, i_k be the list of its distinct elements, arranged so that $m(i_1) \geq \dots \geq m(i_k)$, where $m(i_j)$ is the multiplicity of i_j in M . The *type* of M is the vector $\lambda(M) = (m(i_1), m(i_2), \dots, m(i_k))$. For instance, the multiset $M = \{1, 1, 1, 2, 3, 3\}$ has type $\lambda(M) = (3, 2, 1)$. Multisets are in bijection with monomials of R . Given M , we can construct the monomial:

$$\mathbf{x}_M^{\lambda(M)} = \prod_{j=1}^k x_{i_j}^{m(i_j)}.$$

Conversely, given a monomial, the associated multiset is the set of indices appearing in it, along with multiplicities. The action of \mathfrak{S}_∞ on monomials coincides with the natural action of \mathfrak{S}_∞ on multisets M and this action preserves the type of a multiset (resp. monomial). We also note the following elementary fact.

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Lemma 2. *Let $\sigma \in \mathfrak{S}_\infty$ and $f \in R$. Then there exists a positive integer N and $\tau \in \mathfrak{S}_N$ such that $\tau f = \sigma f$.*

Theorem 1 is a direct corollary of the following result.

Theorem 3. *Let $G = \{g_1, \dots, g_n\}$ be a set of monomials of degree d with distinct types and fix a matrix $C = (c_{ij}) \in K^{n \times n}$ of rank r . Then the submodule $I = \langle f_1, \dots, f_n \rangle_{R[\mathfrak{S}_\infty]} \subseteq R$ generated by the n polynomials, $f_j = \sum_{i=1}^n c_{ij} g_i$, ($j = 1, \dots, n$), cannot have fewer than r $R[\mathfrak{S}_\infty]$ -generators.*

Proof. Suppose that p_1, \dots, p_k are generators for I ; we prove that $k \geq r$. Since each $p_l \in I$, it follows that each is a linear combination, over $R[\mathfrak{S}_\infty]$, of monomials in G . Therefore, each monomial occurring in p_l has degree at least d , and, moreover, any degree d monomial in p_l has the same type as one of the monomials in G .

Write each of the monomials in G in the form $g_i = \mathbf{x}_{M_i}^{\lambda_i}$ for multisets M_1, \dots, M_n with corresponding distinct types $\lambda_1, \dots, \lambda_n$, and express each generator p_l as:

$$(1) \quad p_l = \sum_{i=1}^n \sum_{\lambda(M)=\lambda_i} u_{ilM} \mathbf{x}_M^{\lambda_i} + q_l,$$

in which $u_{ilM} \in K$ with only finitely many of them nonzero, each monomial in q_l has degree larger than d , and the inner sum is over multisets M with type λ_i .

Since each polynomial in $\{f_1, \dots, f_n\}$ is a finite linear combination of the p_l , and since only finitely many integers are indices of monomials appearing in p_1, \dots, p_k , we may pick N large enough so that all of these linear combinations can be expressed with coefficients in the subring $R[\mathfrak{S}_N]$ (c.f. Lemma 2). Therefore, we have,

$$(2) \quad f_j = \sum_{l=1}^k \sum_{\sigma \in \mathfrak{S}_N} s_{lj\sigma} \sigma p_l,$$

for some polynomials $s_{lj\sigma} \in R$. Substituting (1) into (2) gives us that

$$f_j = \sum_{l=1}^k \sum_{\sigma \in \mathfrak{S}_N} \sum_{i=1}^n \sum_{\lambda(M)=\lambda_i} v_{lj\sigma} u_{ilM} \mathbf{x}_{\sigma M}^{\lambda_i} + h_j,$$

in which each monomial appearing in $h_j \in R$ has degree greater than d and $v_{lj\sigma}$ is the constant term of $s_{lj\sigma}$. Since each f_j has degree d , we have that $h_j = 0$. Thus,

$$\sum_{i=1}^n c_{ij} \mathbf{x}_{M_i}^{\lambda_i} = \sum_{l=1}^k \sum_{\sigma \in \mathfrak{S}_N} \sum_{i=1}^n \sum_{\lambda(M)=\lambda_i} v_{lj\sigma} u_{ilM} \mathbf{x}_{\sigma M}^{\lambda_i}.$$

Next, for a fixed i , take the sum on each side in this last equation of the coefficients of monomials with the type λ_i . This produces the n^2 equations:

$$c_{ij} = \sum_{l=1}^k \sum_{\sigma \in \mathfrak{S}_N} \sum_{\lambda(M)=\lambda_i} v_{lj\sigma} u_{ilM} = \sum_{l=1}^k \left(\sum_{\lambda(M)=\lambda_i} u_{ilM} \right) \left(\sum_{\sigma \in \mathfrak{S}_N} v_{lj\sigma} \right) = \sum_{l=1}^k U_{il} V_{lj},$$

in which $U_{il} = \sum_{\lambda(M)=\lambda_i} u_{ilM}$ and $V_{lj} = \sum_{\sigma \in \mathfrak{S}_N} v_{lj\sigma}$. Set U to be the $n \times k$ matrix (U_{il}) and similarly let V denote the $k \times n$ matrix (V_{lj}) . These n^2 equations are represented by the equation $C = UV$, leading to the following chain of inequalities:

$$r = \text{rank}(C) = \text{rank}(UV) \leq \min\{\text{rank}(U), \text{rank}(V)\} \leq \min\{n, k\} \leq k.$$

Therefore, we have $k \geq r$, and this completes the proof. \square

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