

# MINIMAL GENERATORS FOR SYMMETRIC IDEALS

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ABSTRACT. Let  $R = K[X]$  be the polynomial ring in infinitely many indeterminates  $X$  over a field  $K$ , and let  $\mathfrak{S}_X$  be the symmetric group of  $X$ . The group  $\mathfrak{S}_X$  acts naturally on  $R$ , and this in turn gives  $R$  the structure of a module over the group ring  $R[\mathfrak{S}_X]$ . A recent theorem of Aschenbrenner and Hillar states that the module  $R$  is Noetherian. We address whether submodules of  $R$  can have any number of minimal generators, answering this question positively.

Let  $R = K[X]$  be the polynomial ring in infinitely many indeterminates  $X$  over a field  $K$ . Write  $\mathfrak{S}_X$  (resp.  $\mathfrak{S}_N$ ) for the symmetric group of  $X$  (resp.  $\{1, \dots, N\}$ ) and  $R[\mathfrak{S}_X]$  for its (left) group ring, which acts naturally on  $R$ . A *symmetric ideal*  $I \subseteq R$  is an  $R[\mathfrak{S}_X]$ -submodule of  $R$ .

Aschenbrenner and Hillar recently proved [1] that all symmetric ideals are finitely generated over  $R[\mathfrak{S}_X]$ . They were motivated by finiteness questions in chemistry [2] and algebraic statistics [4]. In proving the Noetherianity of  $R$ , it was shown that a symmetric ideal  $I$  has a special, finite set of generators called a *minimal Gröbner basis*. However, the more basic question whether  $I$  is always cyclic (already asked by Josef Schicho [3]) was left unanswered in [1]. Our result addresses a generalization of this important issue.

**Theorem 1.** *For every positive integer  $n$ , there are symmetric ideals of  $R$  generated by  $n$  polynomials which cannot have fewer than  $n$   $R[\mathfrak{S}_X]$ -generators.*

In what follows, we work with the set  $X = \{x_1, x_2, x_3, \dots\}$ , although as remarked in [1], this is not really a restriction. In this case,  $\mathfrak{S}_X$  is naturally identified with  $\mathfrak{S}_\infty$ , the permutations of the positive integers, and  $\sigma x_i = x_{\sigma i}$  for  $\sigma \in \mathfrak{S}_\infty$ .

Let  $M$  be a finite multiset of positive integers and let  $i_1, \dots, i_k$  be the list of its distinct elements, arranged so that  $m(i_1) \geq \dots \geq m(i_k)$ , where  $m(i_j)$  is the multiplicity of  $i_j$  in  $M$ . The *type* of  $M$  is the vector  $\lambda(M) = (m(i_1), m(i_2), \dots, m(i_k))$ . For instance, the multiset  $M = \{1, 1, 1, 2, 3, 3\}$  has type  $\lambda(M) = (3, 2, 1)$ . Multisets are in bijection with monomials of  $R$ . Given  $M$ , we can construct the monomial:

$$\mathbf{x}_M^{\lambda(M)} = \prod_{j=1}^k x_{i_j}^{m(i_j)}.$$

Conversely, given a monomial, the associated multiset is the set of indices appearing in it, along with multiplicities. The action of  $\mathfrak{S}_\infty$  on monomials coincides with the natural action of  $\mathfrak{S}_\infty$  on multisets  $M$  and this action preserves the type of a multiset (resp. monomial). We also note the following elementary fact.

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**Lemma 2.** *Let  $\sigma \in \mathfrak{S}_\infty$  and  $f \in R$ . Then there exists a positive integer  $N$  and  $\tau \in \mathfrak{S}_N$  such that  $\tau f = \sigma f$ .*

Theorem 1 is a direct corollary of the following result.

**Theorem 3.** *Let  $G = \{g_1, \dots, g_n\}$  be a set of monomials of degree  $d$  with distinct types and fix a matrix  $C = (c_{ij}) \in K^{n \times n}$  of rank  $r$ . Then the submodule  $I = \langle f_1, \dots, f_n \rangle_{R[\mathfrak{S}_\infty]} \subseteq R$  generated by the  $n$  polynomials,  $f_j = \sum_{i=1}^n c_{ij} g_i$ , ( $j = 1, \dots, n$ ), cannot have fewer than  $r$   $R[\mathfrak{S}_\infty]$ -generators.*

*Proof.* Suppose that  $p_1, \dots, p_k$  are generators for  $I$ ; we prove that  $k \geq r$ . Since each  $p_l \in I$ , it follows that each is a linear combination, over  $R[\mathfrak{S}_\infty]$ , of monomials in  $G$ . Therefore, each monomial occurring in  $p_l$  has degree at least  $d$ , and, moreover, any degree  $d$  monomial in  $p_l$  has the same type as one of the monomials in  $G$ .

Write each of the monomials in  $G$  in the form  $g_i = \mathbf{x}_{M_i}^{\lambda_i}$  for multisets  $M_1, \dots, M_n$  with corresponding distinct types  $\lambda_1, \dots, \lambda_n$ , and express each generator  $p_l$  as:

$$(1) \quad p_l = \sum_{i=1}^n \sum_{\lambda(M)=\lambda_i} u_{ilM} \mathbf{x}_M^{\lambda_i} + q_l,$$

in which  $u_{ilM} \in K$  with only finitely many of them nonzero, each monomial in  $q_l$  has degree larger than  $d$ , and the inner sum is over multisets  $M$  with type  $\lambda_i$ .

Since each polynomial in  $\{f_1, \dots, f_n\}$  is a finite linear combination of the  $p_l$ , and since only finitely many integers are indices of monomials appearing in  $p_1, \dots, p_k$ , we may pick  $N$  large enough so that all of these linear combinations can be expressed with coefficients in the subring  $R[\mathfrak{S}_N]$  (c.f. Lemma 2). Therefore, we have,

$$(2) \quad f_j = \sum_{l=1}^k \sum_{\sigma \in \mathfrak{S}_N} s_{lj\sigma} \sigma p_l,$$

for some polynomials  $s_{lj\sigma} \in R$ . Substituting (1) into (2) gives us that

$$f_j = \sum_{l=1}^k \sum_{\sigma \in \mathfrak{S}_N} \sum_{i=1}^n \sum_{\lambda(M)=\lambda_i} v_{lj\sigma} u_{ilM} \mathbf{x}_{\sigma M}^{\lambda_i} + h_j,$$

in which each monomial appearing in  $h_j \in R$  has degree greater than  $d$  and  $v_{lj\sigma}$  is the constant term of  $s_{lj\sigma}$ . Since each  $f_j$  has degree  $d$ , we have that  $h_j = 0$ . Thus,

$$\sum_{i=1}^n c_{ij} \mathbf{x}_{M_i}^{\lambda_i} = \sum_{l=1}^k \sum_{\sigma \in \mathfrak{S}_N} \sum_{i=1}^n \sum_{\lambda(M)=\lambda_i} v_{lj\sigma} u_{ilM} \mathbf{x}_{\sigma M}^{\lambda_i}.$$

Next, for a fixed  $i$ , take the sum on each side in this last equation of the coefficients of monomials with the type  $\lambda_i$ . This produces the  $n^2$  equations:

$$c_{ij} = \sum_{l=1}^k \sum_{\sigma \in \mathfrak{S}_N} \sum_{\lambda(M)=\lambda_i} v_{lj\sigma} u_{ilM} = \sum_{l=1}^k \left( \sum_{\lambda(M)=\lambda_i} u_{ilM} \right) \left( \sum_{\sigma \in \mathfrak{S}_N} v_{lj\sigma} \right) = \sum_{l=1}^k U_{il} V_{lj},$$

in which  $U_{il} = \sum_{\lambda(M)=\lambda_i} u_{ilM}$  and  $V_{lj} = \sum_{\sigma \in \mathfrak{S}_N} v_{lj\sigma}$ . Set  $U$  to be the  $n \times k$  matrix  $(U_{il})$  and similarly let  $V$  denote the  $k \times n$  matrix  $(V_{lj})$ . These  $n^2$  equations are represented by the equation  $C = UV$ , leading to the following chain of inequalities:

$$r = \text{rank}(C) = \text{rank}(UV) \leq \min\{\text{rank}(U), \text{rank}(V)\} \leq \min\{n, k\} \leq k.$$

Therefore, we have  $k \geq r$ , and this completes the proof.  $\square$

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