Chapter 3

Logarithmic Derivatives

3.1 Statement of Results

Using Gröbner basis techniques, we provide new constructive proofs of two theorems of Harris and Sibuya [21, 22] (see also, [60, 61] and [62, Problem 6.60]) that give degree bounds and allow for several generalizations. To prepare for the statement of the result, we begin with some preliminary definitions.

Definition 3.1.1. A *differential field* is a field K equipped with a map called a *derivation* $D : K \to K$ that is linear and satisfies the ordinary rule for derivatives; i.e.,

$$D(u+v) = D(u) + D(v), \quad D(uv) = uD(v) + vD(u).$$

When it is more convenient, we sometimes write u', u'', etc. for Du, D^2u, etc. Let F be a differential field extension of K (that is, a field extension that is also a differential field). A *linear homogeneous differential polynomial* $L(Y)$ over K of order m is a mapping from F to itself of form

$$L(Y) = a_m D^m(Y) + a_{m-1} D^{m-1}(Y) + \cdots + a_1 D(Y) + a_0 Y, \quad a_i \in K, a_m \neq 0.$$

We may now state the results of Harris and Sibuya.

Proposition 3.1.2. *Let $N_1, N_2 > 1$ be positive integers and let K be a differential field of characteristic 0. Let F be a (differential) field extension of K and suppose that $L_1(Y)$*

and $L_2(Y)$ are nonzero homogeneous linear differential polynomials (of orders N_1 and N_2 respectively) with coefficients in K. Further, suppose that one of the following holds:

(1) *$y \in F$ has $L_1(y) = L_2(1/y) = 0$, or*

(2) *$N_2 \leq q \in \mathbb{Z}_+$, and $y \in F$ has $L_1(y) = L_2(y^q) = 0$.*

Then, Dy/y is algebraic over K.

In this chapter, we prove the following more refined result.

Theorem 3.1.3. *Let $N_1, N_2 > 1$ be positive integers and let K be a differential field of characteristic 0. Let F be a (differential) field extension of K. Suppose that $L_1(Y)$ and $L_2(Y)$ are nonzero homogeneous linear differential polynomials (of orders N_1 and N_2 respectively) with coefficients in K. Further, suppose that one of the following holds:*

(1) *$y \in F$ has $L_1(y) = L_2(1/y) = 0$, or*

(2) *$N_2 \leq q \in \mathbb{Z}_+$, and $y \in F$ has $L_1(y) = L_2(y^q) = 0$.*

Then, $D^j y/y$ is algebraic over K for all $j \geq 1$. Moreover, the degree of the minimal polynomial for $D^j y/y$ ($j = 1, \ldots, N_1 - 1$) in (1) is at most $\binom{N_2+N_1-2}{N_1-1}$.

Remark 3.1.4. We note that with a more careful analysis, one may use our techniques to get similar results for fields of sufficiently large characteristic.

The first part (algebraicity) of this theorem is proved in Section 3.3, while in Section 3.4, we prove the specified degree bounds. Finally, in Section 3.5, we describe how our technique applies to certain nonlinear differential equations. Recall that a polynomial $f \in K[x]$ is called *separable* if all of its roots are distinct, and a field K is called *perfect* if every irreducible polynomial in $K[x]$ is separable. Examples of perfect fields include finite fields, fields of characteristic zero, and, of course, algebraically closed fields. It is interesting to note that there is a converse to Theorem 3.1.3 for this class of fields.

Proposition 3.1.5. *Let K be a perfect field. If y'/y is algebraic over K, then both y and $1/y$ satisfy linear differential equations over K.*

Proof. Suppose that K is perfect and $u = y'/y$ is algebraic over K. Let $f(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0 \in K[x]$ be the monic, irreducible polynomial for u over K. Since K is perfect, it follows from basic field theory that $\gcd(f, \frac{\partial f}{\partial x}) = 1$. In particular, $\frac{\partial f}{\partial x} \neq 0$. Consider now,

$$0 = f(u)' = u'\left(mu^{m-1} + \sum_{i=1}^{m-1} ia_i u^{i-1}\right) + \sum_{i=0}^{m-1} a_i' u^i.$$

Since $\frac{\partial f}{\partial x} = mx^{m-1} + \sum_{i=1}^{m-1} ia_i x^{i-1}$ is not the zero polynomial, it follows from the irreducibility of f that $mu^{m-1} + \sum_{i=1}^{m-1} ia_i u^{i-1} \neq 0$. Hence, $u' \in K(u)$ and the same holds for higher derivatives.

Next, notice that $(1/y)' = -y'/y^2 = -u/y$ and an easy induction gives us that $(1/y)^{(k)} = p_k(u, u', u'', \ldots)/y$, in which p_k is a polynomial (over K) in u and its derivatives (set $p_0 = 1$). By above, the polynomials $p_k(u, u', \ldots)$ lie in the field $K(u)$. This implies that they satisfy some (non-trivial) linear dependence relation,

$$\sum_{k=0}^{N} h_k p_k = 0,$$

in which $h_k \in K$. Therefore,

$$0 = \sum_{k=0}^{N} h_k p_k / y = \sum_{k=0}^{N} h_k (1/y)^{(k)}$$

as desired. Performing a similar examination on the derivatives of $y' = uy$ produces a linear differential equation for y over K, completing the proof. $\square$

As an application of our main theorem, take F to be the field of complex meromorphic functions on $\mathbb{C}$ and $K = \mathbb{Q}$. Then, the only y such that both y and $1/y$ satisfy linear differential equations over K are the functions, $y = ce^{ux}$, in which u is an algebraic number of degree at most $\min\{N_1, N_2\}$ and $c \in \mathbb{C} \setminus \{0\}$. This simple example shows that it is possible to produce a minimum degree of $\min\{N_1, N_2\}$ for y'/y; however, it is still an open question of whether we can achieve a minimum degree close to the bound given in Theorem 3.1.3.

Theorem 3.1.3 can also be used to show that elements in a differential field F do not satisfy linear differential equations over a subfield K, as the following example demonstrates.

Example 3.1.6. ([62, Problem 6.59]). Let $K = \mathbb{C}(x)$ and $F = \mathbb{C}((x))$. Then, $\sec(x)$ does not satisfy a linear differential equation over K. To see this, suppose otherwise. Then, since $y = \cos(x)$ satisfies a linear differential equation, Theorem 3.1.3 would imply that $y'/y = \cos(x)'/\cos(x) = -\tan(x)$ is algebraic over $\mathbb{C}(x)$, a contradiction.

3.2 Algebraic Preliminaries

We begin by quickly reviewing some standard terminology (some of this material overlaps that of Chapter 1). Let K be a field. A *term order* (or *monomial ordering*) on $\mathbb{N}^n$ is a total order $\prec$ that is a well-ordering and is linear:

$$\boldsymbol{a} \prec \boldsymbol{b} \Rightarrow \boldsymbol{a} + \boldsymbol{c} \prec \boldsymbol{b} + \boldsymbol{c},$$

for $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{N}^n$. This ordering of $\mathbb{N}^n$ gives a corresponding ordering on the monomials of $R = K[x_1, \ldots, x_n]$.

Given a polynomial $f \in R$, the *leading monomial* of f (simply written $\mathrm{lm}_\prec(f)$) is the largest monomial occurring in f with respect to $\prec$. The *initial ideal* of an ideal $I \subseteq R$ is defined to be

$$\mathrm{in}_\prec(I) := \langle \mathrm{lm}_\prec(f) : f \in I \rangle.$$

A Gröbner Basis for an ideal $I \subseteq R$ is a finite subset $G = \{g_1, \ldots, g_m\}$ of I such that:

$$\langle \mathrm{lm}_\prec(g_1), \ldots, \mathrm{lm}_\prec(g_m) \rangle = \mathrm{in}_\prec(I).$$

There is a canonical Gröbner basis for an ideal with respect to a fixed term order called the *reduced Gröbner Basis* of I, and it can be computed algorithmically [9].

Let I be an ideal of a polynomial ring $R = K[x_1, \ldots, x_n]$ over the field K and let $V(I)$ be the corresponding variety (we work over $\overline{K}^n$ to simplify exposition):

$$V(I) := \{(a_1, \ldots, a_n) \in \overline{K}^n : f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\}.$$

We call $V(I)$ *zero-dimensional* if it consists of a finite number of points. The following characterization of zero-dimensional varieties can be found in [9, p. 230].

Theorem 3.2.1. *Let $V(I)$ be a variety in $\overline{K}^n$ and fix a term ordering $\prec$ for $K[x_1, \ldots, x_n]$. Then the following statements are equivalent:*

(1) *V is a finite set.*

(2) *For each i, $1 \leq i \leq n$, there is some $m_i \geq 0$ such that $x_i^{m_i} \in \mathrm{in}_{\prec}(I)$.*

(3) *Let G be a Gröbner basis for I. Then for each i, $1 \leq i \leq n$, there is some $m_i \geq 0$ such that $x_i^{m_i} = \mathrm{lm}_{\prec}(g)$ for some $g \in G$.*

The following fact is well-known, but we include a proof for completeness.

Proposition 3.2.2. *If $V(I)$ is a zero-dimensional variety, then the coordinates of every point of $V(I)$ are algebraic over K.*

Proof. Let $(a_1, \ldots, a_n)$ be a point in $V(I)$. We prove that a_1 is algebraic over K (the other coordinates are treated similarly). Fix a lexicographic term order $\prec$ on $K[x_1, \ldots, x_n]$ such that $x_1 < x_2 < \cdots < x_n$, and let G be a reduced Gröbner basis for I with respect to this term order. Then, it follows from Theorem 3.2.1 that $x_1^m = \mathrm{lm}_{\prec}(g)$ for some $0 \neq g \in G$ and $m \geq 0$. Since G is computed using operations in the field K (the ideal I is defined over K), it follows that $g \in K[x_1, \ldots, x_n]$. Moreover, our term order insures that $g(x_1, \ldots, x_n) = g(x_1)$ is a univariate polynomial in the variable x_1. Since $g(a_1, \ldots, a_n) \in I$, we must have that $g(a_1) = 0$. It follows that a_1 is algebraic over K, completing the proof. $\square$

Proposition 3.2.2 is an important tool in the proof of our main theorem. We now describe another ingredient in the solution of our problem, although its generality should be useful in many other contexts. Give R a grading by assigning to each x_i, a number $w(x_i) = w_i \in \mathbb{N}$, so that

$$w\left(\prod_{i=1}^{n} x_i^{v_i}\right) = \sum_{i=1}^{n} v_i w_i.$$

Then, we have the following extension of a result of Sperber [61]. A proof of a generalization can be found in [57, Lemma 2.2.2]; however, again for completeness we include an argument for our special case.

Lemma 3.2.3. *Let I be the ideal of $R = K[x_1, \ldots, x_n]$ generated by a collection of polynomials, $\{f_\beta\}_{\beta\in\Gamma} \subseteq R$. Let $\tilde{f}_\beta$ be the leading homogeneous form of f_β with respect to the above grading, and let J be the ideal generated by $\{\tilde{f}_\beta\}_{\beta\in\Gamma}$. Then, if $V(J)$ is zero-dimensional, so is $V(I)$.*

Proof. Fix a grading $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$ and let $\prec$ be a monomial ordering on R. Define a new monomial ordering $\prec_w$ as follows [64, p. 4]: for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^n$ we set

$$\boldsymbol{a} \prec_w \boldsymbol{b} \ :\Leftrightarrow \ w \cdot \boldsymbol{a} < w \cdot \boldsymbol{b} \ \text{ or } \ (w \cdot \boldsymbol{a} = w \cdot \boldsymbol{b} \text{ and } \boldsymbol{a} \prec \boldsymbol{b}).$$

Since $V(J)$ is zero-dimensional, as before, Theorem 3.2.1 tells us that for each $i \in \{1, \ldots, n\}$ there exist integers $m_i \geq 0$ such that $x_i^{m_i} \in \mathrm{in}_{\prec_w}(J)$. From Dickson's Lemma [9, p. 69], it follows that $\mathrm{in}_{\prec_w}(J)$ can be finitely generated as

$$\langle \mathrm{lm}_{\prec_w}(\tilde{f}_{\beta_1}), \ldots, \mathrm{lm}_{\prec_w}(\tilde{f}_{\beta_q}) \rangle$$

for some positive integer q and $\beta_j \in \Gamma$. Thus, we may write

$$x_i^{m_i} = \sum_{j=1}^{q} g_{i,j} \cdot \mathrm{lm}_{\prec_w}(\tilde{f}_{\beta_j})$$

for polynomials $g_{i,j}$. Set $\tilde{g}_{i,j}$ to be the terms in $g_{i,j}$ of weight $m_i w_i - w(\tilde{f}_{\beta_j})$, and also let $\hat{g}_{i,j} = g_{i,j} - \tilde{g}_{i,j}$. Notice that the equation above then implies

$$x_i^{m_i} = \sum_{j=1}^{q} \hat{g}_{i,j} \cdot \mathrm{lm}_{\prec_w}(\tilde{f}_{\beta_j}) + \sum_{j=1}^{q} \tilde{g}_{i,j} \cdot \mathrm{lm}_{\prec_w}(\tilde{f}_{\beta_j}).$$

The first sum on the right above has terms of weight that are different from $m_i w_i$, while the second has terms of only this weight. Since the left-hand-side of the equation has weight $m_i w_i$, we must have that

$$\sum_{j=1}^{q} \hat{g}_{i,j} \cdot \mathrm{lm}_{\prec_w}(\tilde{f}_{\beta_j}) = 0.$$

Finally, define

$$h_i = \sum_{j=1}^{q} \tilde{g}_{i,j} f_{\beta_j} \ \in \ I.$$

It is clear that the leading term (with respect to $\prec_w$) of h_i is $x_i^{m_i}$. But then again using Theorem 3.2.1, we have that $V(I)$ is a finite set, completing the proof. $\square$

In other words, this lemma says that in many instances information about an ideal I can be uncovered by passing to a simpler ideal involving leading forms. This fundamental concept is an important component in Gröbner deformation theory.

3.3 Proofs of The Main Theorems

Before embarking on proofs of the theorems stated in Section 3.1, we present a simple example to illustrate our technique. Let $y_1, y_2, \ldots$ be variables. We will view $y_j = D^j y/y$ as solutions to a system of polynomial equations over $K[\{y_j\}_{j=1}^{\infty}]$. For example, consider the system ($N_1 = 3, N_2 = 2$):

$$y''' + a_2 y'' + a_1 y' + a_0 y = 0,$$
$$(1/y)'' + b_1 (1/y)' + b_0 (1/y) = 0$$

in which $a_2, a_1, a_0, b_1, b_0 \in K$. Dividing the first equation by y and expanding the second one gives us the more suggestive equations:

$$y_3 + a_2 y_2 + a_1 y_1 + a_0 = 0,$$
$$(2y_1^2 - y_2) - b_1 y_1 + b_0 = 0.$$

Also, differentiating the original equation for $1/y$ and expanding, we have that

$$-6y_1^3 + 6y_1 y_2 - y_3 + b_1(2y_1^2 - y_2) - y_1(b_1' + b_0) + b_0' = 0.$$

Thus, we may view $(y'/y, y''/y, y'''/y) = (y_1, y_2, y_3)$ as a solution to a system of three polynomial equations in three unknowns.

Let $w(y_i) = i$ define a grading of $K[y_1, y_2, y_3]$, and notice that the system of leading forms, $\{y_3 = 0, 2y_1^2 - y_2 = 0, -6y_1^3 + 6y_1 y_2 - y_3 = 0\}$, has only the trivial solution $(y_1, y_2, y_3) = (0, 0, 0)$. In light of Lemma 3.2.3, it follows that the equations above define a zero-dimensional variety. Therefore, appealing to Proposition 3.2.2, we have established the algebraicity component of Theorem 3.1.3 (1) for this example ($N_1 = 3, N_2 = 2$).

In general, we will construct a system of $N_1 - 1$ equations in $N_1 - 1$ unknowns satisfied by the y_i. These equations will define a zero-dimensional variety, and thus, standard elimination techniques (see, for instance, [8]) give us a direct method of computing, for each i, a nonzero polynomial (over K) satisfied by y_i.

Let us first examine what happens when we compute $f_n = D^n(1/y)$. Notice that

$$\begin{aligned}
f_0 &= 1/y \\
f_1 &= -y^{-2}Dy = -y_1/y \\
f_2 &= 2y^{-3}(Dy)^2 - y^{-2}D^2y = 2y_1^2/y - y_2/y \\
f_3 &= -6y_1^3/y + 6y_1y_2/y - y_3/y.
\end{aligned}$$

In general, these functions f_n can be expressed in the form $f_n = (1/y)p_n(y_1, \ldots, y_n)$ for polynomials $p_n \in \mathbb{Z}[y_1, \ldots, y_n]$. Moreover, with respect to the grading $w(y_i) = i$, these p_n are homogeneous of degree n. These facts are easily deduced from the following lemma.

Lemma 3.3.1. *Let $m \in \mathbb{Z}_+$. Then,*

$$\frac{p_m}{m!} = -\sum_{j=1}^{m-1} \frac{p_{m-j}}{(m-j)!}\frac{y_j}{j!} - \frac{y_m}{m!}.$$

Proof. Consider the following well-known identity (Leibniz' rule),

$$\sum_{j=0}^{m} \binom{m}{j} \left(D^j h\right)\left(D^{m-j} g\right) = D^m(hg).$$

Setting $h = y$ and $g = 1/y$, it follows that

$$\sum_{j=0}^{m} \frac{D^j y}{j!}\frac{D^{m-j}(1/y)}{(m-j)!} = 0.$$

Multiplying the numerator and denominator by y and rewriting this expression gives us

$$\frac{p_m}{m!} = -\frac{p_{m-1}}{(m-1)!}\frac{y_1}{1!} - \frac{p_{m-2}}{(m-2)!}\frac{y_2}{2!} - \cdots - \frac{p_1}{1!}\frac{y_{m-1}}{(m-1)!} - \frac{y_m}{m!}.$$

□

We are now ready to prove Theorem 3.1.3 (1).

Proof of Theorem 3.1.3 (1). With N_1, N_2 as in Theorem 3.1.3, we suppose $N_1 = n$, $N_2 = m$. Dividing through by y in the first differential equation for y gives us

$$y_n = -a_{n-1}y_{n-1} - \cdots - a_1y_1 - a_0, \quad a_i \in K \tag{3.3.1}$$

while multiplying the second one for $1/y$ by y produces the equation

$$p_m + b_{m-1}p_{m-1} + \cdots + b_0 = 0, \qquad b_i \in K.$$

Differentiating k times the original linear differential equation for y, we will arrive at linear equations $y_{n+k} = L_k(y_1, \ldots, y_{n-1})$ in terms (over K) of $y_1, \ldots, y_{n-1}$ like (3.3.1) above (by repeated substitution of the previous linear equations). If we also differentiate the equation for $1/y$ k times, we will produce another equation for the variables y_i. More formally, we have that

$$D^{m+k}(1/y) + D^k(b_{m-1}D^{m-1}(1/y)) + \cdots + D^k(b_0/y) = 0$$

produces the equation (by Leibniz' rule)

$$D^{m+k}(1/y) + \sum_{i=0}^{m-1}\sum_{j=0}^{k}\binom{k}{j}\left(D^j b_i\right)\left(D^{k-j+i}(1/y)\right) = 0.$$

So finally (after multiplying through by y), it follows that

$$P_{m+k} := p_{m+k} + \sum_{i=0}^{m-1}\sum_{j=0}^{k}\binom{k}{j}\left(D^j b_i\right)p_{k-j+i} = 0. \tag{3.3.2}$$

It is clear that the leading homogeneous forms of the P_{m+k} (with respect to the grading above) are p_{m+k}. Consider now the ring homomorphism $\phi : K[\{y_i\}_{i=1}^{\infty}] \to K[y_1, \ldots, y_{n-1}]$ defined by sending $y_j \mapsto 0$ for $j \geq n$ and $y_j \mapsto y_j$ for $j < n$. Let $\tilde{P}_{m+k}$ denote the polynomials produced by substituting the linear forms L_i for the variables y_{n+i} $(i = 0, 1, \ldots)$ into the polynomials, P_{m+k}. The leading homogeneous forms of the $\tilde{P}_{m+k}$ will just be $\tilde{p}_{m+k} := \phi(p_{m+k})$ because we are substituting linear polynomials with strictly smaller degree (corresponding to the grading). In light of Lemma 3.2.3, we verify that the $n-1$ equations (in the $n-1$ variables),

$$\tilde{p}_m = 0,\ \tilde{p}_{m+1} = 0,\ \ldots,\ \tilde{p}_{m+n-2} = 0, \tag{3.3.3}$$

are only satisfied by the point $(0, \ldots, 0)$ to prove the claim.

Suppose that $(y_1, \ldots, y_{n-1}) \neq (0, \ldots, 0)$ is a zero of the system in (3.3.3); we will derive a contradiction. Let $r \in \{1, \ldots, n-1\}$ be the largest integer such that $y_r \neq 0$, and choose $t \in \{0, \ldots, m-1\}$ maximal such that $\tilde{p}_{m-t} = 0$, $\tilde{p}_{m-t+1} = 0$, …, $\tilde{p}_m = 0$. If $t = m-1$, then $\tilde{p}_1 = -y_1 = 0$, and so the recurrence in Lemma 3.3.1 and (3.3.3) give us that $y_i = 0$ for $i \in \{1, \ldots, n-1\}$, a contradiction. Thus, $t \le m-2$. Using Lemma 3.3.1 with ϕ (and the maximality of r), we have the following identity:

$$\frac{\tilde{p}_{m-t+r-1}}{(m-t+r-1)!} = -\frac{\tilde{p}_{m-t+r-2}}{(m-t+r-2)!}\frac{y_1}{1!} - \cdots - \frac{\tilde{p}_{m-t}}{(m-t)!}\frac{y_{r-1}}{(r-1)!} - \frac{\tilde{p}_{m-t-1}}{(m-t-1)!}\frac{y_r}{r!}.$$

From (3.3.3) and the property of t above, it follows that $\frac{\tilde{p}_{m-t-1}}{(m-t-1)!}\frac{y_r}{r!} = 0$. Thus, $y_r = 0$ or $\tilde{p}_{m-(t+1)} = 0$; the first possibility contradicts $y_r \neq 0$, while the second contradicts maximality of t.

This proves that the equations (3.3.3) define a zero-dimensional variety, from which the algebraicity of $D^j y/y$ ($j = 1, \ldots, n-1$) follows using Proposition 3.2.2. With repeated differentiation of (3.3.1), we also see that $D^j y/y$ is algebraic for all $j \ge n$. The proof of the degree bounds will be postponed until Section 3.4. $\square$

The proof for Theorem 3.1.3 (2) is similar to the one above, however, the recurrences as in Lemma 3.3.1 are somewhat more complicated. Let $n \in \mathbb{N}$, $q \in \mathbb{Z}_+$ and examine $f_{n,q} = D^n(y^q)$. It turns out that $f_{n,q} = y^q p_{n,q}(y_1, \ldots, y_n)$ in which $p_{n,q} \in \mathbb{Z}[y_1, \ldots, y_n]$ is homogeneous of degree n (with respect to the grading $w(y_i) = i$). This follows in a similar manner as before from the following lemma.

Lemma 3.3.2. *Let $p_{n,1} = y_n$ for $n \in \mathbb{N}$ ($y_0 = 1$). Then, for all $m \in \mathbb{N}, q > 1$,*

$$p_{m,q} = y_m + \sum_{j=0}^{m-1} \binom{m}{j} y_j p_{m-j,q-1}.$$

Proof. Use Leibniz' rule as in Lemma 3.3.1 with $h = y^{q-1}$ and $g = y$. $\square$

The next lemma will be used in the proof of Theorem 3.1.3 (2), and it follows from a straightforward induction on a (using Lemma 3.3.2).

Lemma 3.3.3. *Let ϕ be as in the proof of Theorem 3.1.3 (1) and $n \ge 2$. Then, for all $a \in \mathbb{Z}_+$ and $b \in \mathbb{N}$, we have $\phi\left(p_{(a+1)(n-1)+b,a}\right) = 0$.*

We now prove Theorem 3.1.3 (2).

Proof of Theorem 3.1.3 (2). With N_1, N_2 as in Theorem 3.1.3, we suppose $N_1 = n$, $N_2 = m \leq q$. As before, the first differential equation for y gives us

$$y_n = -a_{n-1}y_{n-1} - \cdots - a_1y_1 - a_0 \quad a_i \in K \tag{3.3.4}$$

while the second one for y^q (after dividing through by y^q) produces the equation

$$p_{m,q} + b_{m-1}p_{m-1,q} + \cdots + b_0 = 0 \quad b_i \in K.$$

Differentiating k times the original linear differential equation for y, produces linear equations $y_{n+k} = L_k(y_1, \ldots, y_{n-1})$ in terms (over K) of $y_1, \ldots, y_{n-1}$ like (3.3.4) above. If we also differentiate the equation for y^q, k times, we will arrive at another equation for the variables y_i:

$$P_{m+k,q} := p_{m+k,q} + \sum_{i=0}^{m-1}\sum_{j=0}^{k}\binom{k}{j}\left(D^jb_i\right)p_{k-j+i,q} = 0.$$

It is clear that the leading homogeneous forms of the $P_{m+k,q}$ (with respect to the grading above) are $p_{m+k,q}$. Let ϕ be as in the proof of Theorem 3.1.3 (1), and let $\tilde{P}_{m+k,q}$ denote the polynomials produced by substituting the linear forms L_i for the variables y_{n+i} $(i = 0, 1, \ldots)$ into the polynomials, $P_{m+k,q}$. If $\tilde{p}_{m+k,q} := \phi(p_{m+k,q}) \neq 0$, then the leading homogeneous form of $\tilde{P}_{m+k,q}$ is $\tilde{p}_{m+k,q}$ because we are substituting linear polynomials with strictly smaller degree (corresponding to the grading).

Consider the following system of equations (recall that $q \geq m$ and $n \geq 2$),

$$\tilde{p}_{m,q} = 0,\ \tilde{p}_{m+1,q} = 0,\ \ldots,\ \tilde{p}_{(q+1)(n-1)-1,q} = 0. \tag{3.3.5}$$

We claim that $(0, \ldots, 0)$ is the only solution to (3.3.5). Suppose, on the contrary, that $(y_1, \ldots, y_{n-1}) \neq (0, \ldots, 0)$ is a solution to (3.3.5), and let $r \in \{1, \ldots, n-1\}$ be the largest integer such that $y_r \neq 0$. Also, choose $t \in \{1, \ldots, q\}$ minimial such that

$$\tilde{p}_{tr,t} = 0,\ \tilde{p}_{tr+1,t} = 0,\ \ldots,\ \tilde{p}_{(t+1)r-1,t} = 0. \tag{3.3.6}$$

Clearly $t \neq 1$, as then $\tilde{p}_{r,1} = y_r = 0$, a contradiction. Applying Lemma 3.3.2 with ϕ (and maximality of r), examine the equation,

$$\tilde{p}_{(t+1)r-1,t} = \tilde{p}_{(t+1)r-1,t-1} + \cdots + \binom{(t+1)r-1}{r}y_r\tilde{p}_{tr-1,t-1}. \tag{3.3.7}$$

Using Lemma 3.3.3 (with $a = t-1$) and the maximality of r, we have $\tilde{p}_{tr+b,t-1} = 0$ for all $b \in \mathbb{N}$. Consequently, (3.3.7) and (3.3.6) imply that $\tilde{p}_{tr-1,t-1} = 0$. Repeating this examination with $\tilde{p}_{(t+1)r-2,t}, \tilde{p}_{(t+1)r-3,t}, \ldots, \tilde{p}_{tr,t}$ (in that order) in place of $\tilde{p}_{(t+1)r-1,t}$ on the left-hand side of (3.3.7), it follows that $\tilde{p}_{tr-i,t-1} = 0$ for $i = 1, \ldots, r$. This, of course, contradicts the minimality of t and proves the claim.

It now follows from Lemma 3.2.3 that the variety determined by

$$\left\{\tilde{P}_{m,q} = 0, \ldots, \tilde{P}_{(q+1)(n-1)-1,q} = 0\right\}$$

is zero-dimensional. An application of Proposition 3.2.2 completes the proof. $\square$

3.4 The Degree Bounds

In this section, we outline how to obtain the degree bounds in Theorem 3.1.3. We begin by stating a useful theorem that bounds the cardinality of a variety by the product of the degrees of the polynomials defining it (see [59] for more details).

Theorem 3.4.1 (Bezout's theorem)**.** *Let K be an arbitrary field, and let $f_1, \ldots, f_t \in K[y_1, \ldots, y_t]$. If $V(f_1, \ldots, f_t)$ is finite, then*

$$|V(f_1, \ldots, f_t)| \leq \prod_{i=1}^{t} \deg(f_i).$$

We next make the following straightforward observation.

Lemma 3.4.2. *Let K be a perfect field, and let $I \subset K[y_1, \ldots, y_t]$ be such that $V(I)$ is finite. Then, the degree of the minimal polynomial for each component of an element in $V(I)$ is bounded by the number of elements of $V(I)$.*

Proof. Suppose that $g(x) \in K[x]$ is the irreducible polynomial for $y \in \overline{K}$, a component of $(y_1, \ldots, y, \ldots, y_t) \in V(I)$. Since K is perfect, this polynomial has distinct roots. Thus, there are $\deg(g)$ distinct embeddings $\sigma : K(y) \to \overline{K}$ that are the identity on K. Moreover, each of these homomorphisms extends to an embedding $\tilde{\sigma} : \overline{K} \to \overline{K}$ [39, p. 233]. In particular, the $\deg(g)$ points, $(\tilde{\sigma}y_1, \ldots, \tilde{\sigma}y, \ldots, \tilde{\sigma}y_t)$, are all distinct elements of $V(I)$. Thus, we must have

$$\deg(g) \leq |V(I)|.$$

This completes the proof. $\square$

Theorem 3.4.3. *Assuming the hypothesis as in Theorem 3.1.3, the degree of the polynomial for $D^j y/y$ ($j = 1, \ldots, N_1 - 1$) over K in (1) is at most $\binom{N_2+N_1-2}{N_1-1}$.*

Proof. Let $N_1 = n$, $N_2 = m$ and set $\tilde{P}_{m+k} \in K[y_1, \ldots, y_{n-1}]$ ($k = 0, \ldots, n-2$) to be the polynomials in (3.3.2) after substitution of the linear forms, $y_{n+i} = L_i(y_1, \ldots, y_{n-1})$. Corresponding to the grading $w(y_j) = j$, the weight of each monomial in $\tilde{P}_{m+k}$ is less than or equal to $m + k$. Let $\tilde{S}$ be the set of all solutions with coordinates in $\overline{K}$ to the system $\{\tilde{P}_{m+k} = 0\}_{k=0}^{n-2}$. Our first goal is to bound the cardinality of $\tilde{S}$ by $\binom{m+n-2}{n-1}$.

Suppose that $\{y_{i,1}, \ldots, y_{i,s}\}$ is the list of all s distinct i-th coordinates of members of $\tilde{S}$. Since K is infinite, there exists $k_i \in K$ such that $y_{i,j} \neq k_i$ for $j = 1, \ldots, s$. Now, let $x_1, \ldots, x_{n-1}$ be variables and consider the new polynomials $F_{m+k} \in K[x_1, \ldots, x_{n-1}]$ produced by the substitution $y_i = x_i^i + k_i$ in the $\tilde{P}_{m+k}$. As the $n-1$ equations $\tilde{P}_{m+k} = 0$ define a zero-dimensional variety, so do the $n-1$ equations $F_{m+k} = 0$.

Let S denote the set of all solutions with coordinates in $\overline{K}$ to the system $\{F_{m+k} = 0\}_{k=0}^{n-2}$. Since the total degree of each F_{m+k} is just $m + k$, we have by Bezout's theorem (Theorem 3.4.1),

$$|S| \leq \frac{(m+n-2)!}{(m-1)!} = (n-1)!\binom{m+n-2}{n-1}.$$

Consider the (set-theoretic) map $\psi : S \to \tilde{S}$ given by

$$(x_1, \ldots, x_{n-1}) \mapsto (x_1 + k_1, \ldots, x_{n-1}^{n-1} + k_{n-1}).$$

It is easy to see that

$$\sum_{s \in \tilde{S}} \left|\psi^{-1}(s)\right| = |S|. \tag{3.4.1}$$

Let $(y_1, \ldots, y_{n-1}) \in \tilde{S}$. By our choice of k_i, the polynomial $h_i(x_i) = x_i^i + k_i - y_i$ has precisely i distinct zeroes. These i roots are distinct since characteristic zero implies that $\gcd(h_i, \frac{\partial h_i}{\partial x}) = 1$. Hence, $\left|\psi^{-1}(s)\right| \geq (n-1)!$ for all $s \in \tilde{S}$, and so from

$$|\tilde{S}|(n-1)! \leq |S| \leq \binom{m+n-2}{n-1}(n-1)!,$$

we arrive at the desired bound on $|\tilde{S}|$.

An application of Lemma 3.4.2 now completes the proof. $\square$

We should also note that the proof above generalizes to bound the number of distinct solutions to certain systems of equations. Specifically, we have the following interesting fact.

Theorem 3.4.4. *Let $w(y_j) = j$ be the grading as above and let K be a field of characteristic zero. Let $m \in \mathbb{Z}_+$ and suppose that $\{F_{m+k}(y_1, \ldots, y_{n-1}) = 0\}_{k=0}^{n-2}$ is a zero-dimensional system of polynomial equations over K such that each monomial in F_{m+k} has weight less than or equal to $m + k$. Then, this system will have at most $\binom{m+n-2}{n-1}$ distinct solutions with coordinates in $\overline{K}$.*

In principle, the number of solutions for a generic system with conditions as in Theorem 3.4.4 can be found by a mixed volume computation and Bernstein's Theorem (see [8], for instance). This approach, however, seems difficult to implement.

3.5 Applications to Nonlinear Differential Equations

In the proof of Theorem 3.1.3, it is clear that the important attributes of the recursions as in (3.3.1) are that they reduce the degree and are polynomial in nature. In particular, it was not necessary that they were linear. For example, the system,

$$yy''' + a(y')^2 + by^2 = 0,$$
$$(1/y)'' + c(1/y)' + d(1/y) = 0$$

gives us the recurrence $y_3 + ay_1^2 + b = 0$ (divide the first equation by y^2), which has y_3 expressible as a polynomial in y_1, y_2 with strictly smaller weight. Repeated differentiation of this equation, preserves this property. In general, let $h \in K[z_1, \ldots, z_n]$ be a homogeneous polynomial (with respect to total degree) such that each monomial $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ has

$$\sum_{i=1}^{n} (i-1)\alpha_i < n.$$

If the hypothesis of Theorem 3.1.3 are weakened to allow y to satisfy an equation of the form, $D^n y = h(y, Dy, \ldots, D^{(n-1)}y)$, then the proof applies without change. A generalization along these lines was also considered by Sperber in [61], however, the techniques developed here give us degree bounds just as in Theorem 3.1.3.