UNIQUENESS OF WORD EQUATIONS IN TWO POSITIVE DEFINITE LETTERS

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Abstract. It is shown that the word equation $XBX^3BX = P$ has a unique $n \times n$ positive definite solution X for each pair of $n \times n$ positive definite P and B.

1. Introduction

We describe a recent idea by Jimmie Lawson and Yongdo Lim [3] of using the Reimannian metric for proving unique solvability of word equations. We first recall a classical result.

Theorem 1.1 (Banach's Fixed-Point Theorem)**.** *If M is a complete metric space and $f : M \to M$ is a contraction mapping, then there exists a unique fixed-point for f.*

Theorem 1.2 (Lawson and Lim)**.** *The symmetric word equation $XBX^3BX = P$ has a unique $n \times n$ positive definite solution X for each pair of $n \times n$ positive definite P and B.*

Proof. Let M be the set of $n \times n$ positive definite matrices. Fix $B, P \in M$ and define a map $f : M \to M$ implicitly as the unique positive definite solution $f(X)$ to the equation

$$f(X)Bf(X)Xf(X)Bf(X) = P.$$

We first verify that f is indeed well-defined. Given a positive definite matrix X, the equation $YXY = P$ has a unique solution $Y = X^{-1}\#P$ (see [2]) that involves the arithmetic-geometric mean $C\#D$ for two positive definite matrices C and D:

$$C\#D = D\#C = C^{1/2}(C^{-1/2}DC^{-1/2})^{1/2}C^{1/2}.$$

In turn, given a positive definite Y, the equation, $Y = f(X)Bf(X)$ has a unique solution $f(X) = B^{-1}\#Y$. Collecting these facts, we find that

$$f(X) = B^{-1}\#(X^{-1}\#P)$$

defines f uniquely.

Fixed-points of the map f are in one-to-one correspondence with solutions to the word equation $XBX^3BX = P$. We next verify that f is a contraction mapping with the complete *Reimannian metric* $d(\cdot, \cdot)$ on M (see, for instance, [4]):

$$d(C, D) = \left(\sum_{i=1}^{n} \ln^2 \lambda_i \right)^{1/2},$$

in which λ_i are the eigenvalues of $C^{-1}D$. It is easily verified that $d(\cdot,\cdot)$ is invariant under positive definite congruence and inversion. Additionally, it is known that $d(C^{1/2}, D^{1/2}) \leq \frac{1}{2}d(C,D)$ (see [5]). Thus,

$$d(f(X), f(Y)) \leq \frac{1}{2}d(P\#X^{-1}, P\#Y^{-1}) \leq \frac{1}{4}d(X,Y),$$

and f is a contraction. Finally, we invoke Banach's Fixed-Point Theorem to finish the proof. □

References

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