The classical Nullstellensatz asserts that a reduced affine variety is known by its closed points; algebraically, a prime ideal in an affine ring is the intersection of the maximal ideals containing it. A leading special case of our theorem says that any affine scheme can be distinguished from its subschemes by its closed points with a bounded index of nilpotency; algebraically, an ideal $I$ in an affine ring $A$ may be written as

$$I = \bigcap_{\mathfrak{m} \in \mathcal{N}} (\mathfrak{m}^e + I),$$  

where $\mathcal{N}$ is the set of maximal ideals containing $I$, and $e$ is an integer depending on the degree of nilpotency of $A/I$.

Our theorem might also be thought of as a sharpening of Zariski's Main Lemma on holomorphic functions [4]. Roughly speaking, this lemma asserts that if a regular function $f$ on an irreducible affine variety $V$ vanishes to order $e$ at each of a dense set $\mathcal{N}$ of closed points of $V$, then it vanishes to order $e$ at the generic point; that is, if $P$ is the prime ideal in $k[x_1, \ldots, x_n]$ defining $V$, then

$$\bigcap_{\mathfrak{m} \in \mathcal{N}} \mathfrak{m}^e = P(e),$$  

the $e$th symbolic power of $P$, where the intersection is taken over a dense set of maximal ideals $\mathfrak{m}$ of $k[x_1, \ldots, x_n]$ containing $P$. Of course this implies that, if $I$ is a $P$-primary ideal containing $P^{(e)}$, then

$$I \supset \bigcap_{\mathfrak{m} \in \mathcal{N}} \mathfrak{m}^e;$$

(*), above, is a sharpening that includes (**).

Our proof is related to Zariski's but is simpler than his.

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Throughout this paper, all rings will be commutative and Noetherian, with identity. If $R$ is a ring and $P$ is a prime ideal of $R$, a finitely generated module $M$ is said to be $P$-coprimary if $P$ is the only associated prime of $M$ [2]. The $e$th symbolic power $P^{(e)}$ of $P$ is by definition the inverse image of $P^e$ in $R$ ($P^e \cap R$ if $R$ is a domain), which is the $P$-primary component of $P^e$.

**Results**

**Theorem.** Let $R$ be a ring, and let $P$ be a prime ideal of $R$. Let $\mathcal{N}$ be a set of maximal ideals $m$ containing $P$ such that $R_m/P_m$ is a regular local ring, and such that

$$\bigcap_{m \in \mathcal{N}} m = P.$$

If $M$ is a finitely generated $P$-coprimary module annihilated by $P^e$, then

$$\bigcap_{m \in \mathcal{N}} m^e M = 0.$$

(Note: Since $M$ is $P$-coprimary, $P^e$ annihilates $M$ if and only if $P^{(e)}$ does.)

**Corollary 1 (Zariski's Main Lemma on Holomorphic Functions [4]).** With $R, P,$ and $\mathcal{N}$ as above we have

$$P^{(e)} \supseteq \bigcap_{m \in \mathcal{N}} m^e \supseteq \bigcap_{m \in \mathrm{max \ spec} R, m \supseteq P} m^e.$$

If $R$ is regular, the inclusions may be replaced by equalities.

**Proof.** The first statement, which is the original "Lemma," follows from our theorem with $M = R/P^{(e)}$. For the second statement, it suffices to prove $m^e \supseteq P^{(e)}$ for any $m \supseteq P$; this is the content of a theorem of Nagata [3, p. 143] and Zariski (see [1, Theorem 1]). The next corollary answers a question of B. Wehrfritz which originally motivated this study.

**Corollary 2.** Let $A$ be a ring finitely generated over a field or over the integers, and let $M$ be a finitely generated $A$-module. For sufficiently large $e$, we have

$$\bigcap_{m \in \mathrm{max \ spec} A} m^e M = 0.$$

(In fact, if $0 = \bigcap M_i$ is a primary decomposition of $0 \subseteq M$, with $M/M_i$ $P_i$-coprimary, and $P_i^e(M/M_i) = 0$, then we may take $e$ to be the maximum of the $e_i$).

**Proof.** By [2, Ch. 12 and 13], the regular locus of any domain $A/P$, finitely generated over the integers or a field, is open and therefore dense. Thus we
may apply our Theorem to the coprimary modules $M/M_i$, with the desired result. \[\]

**Remarks.** (1) Clearly, it suffices in Corollary 2 that $A$ be an excellent Hilbert ring.

(2) The hypothesis of the Theorem that the smooth points are dense in max-spec $R$ cannot simply be dropped: there is a 2-dimensional Noetherian regular factorial ring $R$ whose maximal ideals form a countable set, say $\{m_1, m_2, \ldots\}$, such that $\bigcap_i m_i$ is a nonzero, principal ideal $(f)$, whose generator $f$ is in the $i$th power of $m_i$ for all $i$. Setting $\bar{R} = R/(f^2)$, we see that there is no integer $k$ such that the intersection of the $k$th powers of the maximal ideals of $\bar{R}$ is 0.

The example may be constructed as follows: Let $\{X_n\}$, $\{Y_n\}$ be countable families of indeterminates over an algebraically closed field $K$. Set:

\[
\begin{align*}
f_n &= X_n^n - Y_n^{n+1}; \\
I_n &= (f_2 - f_1, \ldots, f_n - f_1) K[X_1, Y_1, \ldots, X_n, Y_n]; \\
S_n &= K[X_1, Y_1, \ldots, X_n, Y_n]/I_n; \\
U_n &= S_n - \bigcup_{i=1}^n (X_i, Y_i) S_n.
\end{align*}
\]

Then $U_n$ is a multiplicatively closed set in $S_n$, and we set $R_n = U_n^{-1}S_n$. There is an obvious injection $R_n \to R_{n+1}$ which is faithfully flat. We set

\[
R = \lim_{\to} R_n,
\]

and let $f$ be the image of $f_n$ in $R$.

One can verify that the maximal ideals of $R$ are precisely the ideals $(X_i, Y_i)$, and that $R$ and $f$ have the properties above (To prove that $R$ is Noetherian, use Cohen's Theorem [3], noting that primes of $R$ are either maximal, and of the form $(X_i, Y_i)$, or of height 1, and thus principal). The ideal $(f)$ is a prime. Note that $R$ is not pseudogeometric; the integral closure of $R/(f)$ is not a finite $R/(f)$-module.

(3) A different approach to the proof of the Theorem could be obtained by proving some kind of "Uniform Artin-Rees Theorem," which we pose as a problem:

**Problem.** Let $R$ be an affine ring, and suppose that $M \subseteq N$ are finitely generated $R$-modules. Is there an integer $k_0$ such that for all $k > k_0$ and all maximal ideals $m$ of $R$

\[
M \cap m^k N = m^{k-k_0}(M \cap m^{k_0}N)\?
Of course remark 2) shows that this could not be true for all rings over an algebraically closed field.

**Proof of the Theorem.** Let \( M_i = P^i M \) for \( 0 \leq i \leq e \). Since \( (M_i/M_{i+1})_P \) is an \( R/P \)-vector space, we can choose an element \( f \in R - P \) such that each \( (M_i/M_{i+1})_f \) is \( (R/P)_f \)-free.

We now claim that for any \( f \in R - P \), it suffices to prove the corresponding Theorem for the ring \( R_f \), the set \( \mathcal{N}_f = \{ mR_f : m \in \mathcal{N}, f \notin m \} \) of maximal ideals of \( R_f \) and the finitely generated \( R_f \)-module \( M_f \). For,

\[
\cap_{m \in \mathcal{N}} m = \cap_{m \in \mathcal{N}} m_{\mathcal{N}_f} - \left( \cap_{m \in \mathcal{N}} m_{\mathcal{N}_f} \right)_f = P_f,
\]

and \( M_f \) is \( P_f \) coprimary, so the hypothesis of the Theorem is satisfied, and, on the other hand \( M \subset M_f \) and

\[
\bigcap_{m \in \mathcal{N}} m^k M \subset \bigcap_{m \in \mathcal{N}} m^k M_f,
\]

so if the latter module is 0, the former is as well.

Thus we may assume that each \( M_i/M_{i+1} \) is \( R/P \)-free from the outset. Under this hypothesis we will show

\[
\cap_{m \in \mathcal{N}} m^k M \cap M_i = \cap_{m \in \mathcal{N}} m^k M_i\quad (***)
\]

for each \( M_i \) and each \( m \in \mathcal{N} \).

Once this is established, the Theorem will follow at once, since if \( x \in M_i \cap \bigcap_{m \in \mathcal{N}} m^i M \), then by (***), \( x \in \bigcap_{m \in \mathcal{N}} m^i M_i \), so \( x + M_{i+1} \subset \bigcap_{m \in \mathcal{N}} m(M_i/M_{i+1}) = 0 \), so \( x \in M_{i+1} \cap \bigcap_{m \in \mathcal{N}} m^i M \), and, continuing in this way, \( x = 0 \).

It remains to prove (***). Because of the behavior of sets of associated primes with respect to exact sequences,

\[
\text{Ass}(M|mM_i) \subset \text{Ass}(M/M_i) \cup \text{Ass}(M_i/mM_i) = \{ P, m \},
\]

So it suffices to prove (***), after localizing at \( m \).

We will now change notation, and write \( R, M, \ldots \) for \( R_m, M_m, \ldots \). Since \( R/P \) is a regular local ring, \( m/P \) is generated by a regular sequence \( x_1, \ldots, x_d \). Lifting these elements to \( x_1, \ldots, x_d \in R \), we see that \( x_1, \ldots, x_d \) is an \( M_i/M_{i+1} \)-regular sequence for each \( i \). It follows at once that \( x_1, \ldots, x_d \) is an \( M_i/M_i \)-regular sequence for each \( i > 0 \), and thus that

\[
(x_1, \ldots, x_d)M \cap M_i = (x_1, \ldots, x_d)M_i.
\]

On the other hand, \( m = P + (x_1, \ldots, x_d) \), so
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\[ m^eM \cap M_i = \left( \sum_{j+k=e} P^j(x_1, \ldots, x_d)^k M \right) \cap M_i \]
\[ \subseteq (P^eM + (x_1, \ldots, x_d)M) \cap M_i \]
\[ = (x_1, \ldots, x_d)M \cap M_i \]
\[ = (x_1, \ldots, x_d)M_i \]
\[ \subseteq mM_i , \]

as required for (***)}. This completes the proof.

REFERENCES

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