Chains of maps between indecomposable modules

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We shall call a sequence of homomorphisms of modules (or objects in some other Abelian category)

\[ \varepsilon : M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{s-1}} M_s \]

a Harada-Sai sequence if each \( M_i \) is an indecomposable \( R \)-module of finite length, no \( f_i \) is an isomorphism, and the composite map \( M_1 \to M_s \) is nonzero. We write \( \lambda(M_i) \) for the length of \( M_i \). Suppose that all the \( \lambda(M_i) \) are bounded by a number \( b \). Fitting’s lemma shows that if all the \( M_i \) were equal and all the \( f_i \) were equal, then the length \( s \) of the sequence \( \varepsilon \) is at most \( b - 1 \). A significant generalization, the Harada-Sai Lemma [4], says that even with distinct \( M_i \) and \( f_i \) the length \( s \) is bounded, this time by \( 2^b - 1 \). For typical applications see Bongartz [1], Gabriel [3], and Ringel [5].

The aim of this note is to sharpen the Harada-Sai Lemma by showing exactly which length sequences \( (\lambda(M_1), \ldots, \lambda(M_s)) \) are possible, and what the length of the image of the composite map \( M_1 \to M_s \), which we call the composite rank of \( \varepsilon \), can be. Here are some simple examples:

- There is a Harada-Sai sequence with all \( \lambda(M_i) = b \) iff \( s \leq 2^b - 1 \), just over half the maximal length of Harada-Sai sequence where all the \( \lambda(M_i) \leq b \).

- There is no Harada-Sai sequence with length sequence \( (2, 3, 3, 4, 4) \).

- There are Harada-Sai sequences with length sequence \( (3, 2, 3, 3, 3) \), but none with the permuted sequences \( (3, 3, 2, 3, 3) \) or \( (2, 3, 3, 3, 3) \).

- There are Harada-Sai sequences with length sequence \( (4, 4, 2, 4, 3, 4) \) and composite rank 1 and 2, but if we increase the length 2 to length 3, producing a length sequence \( (4, 4, 3, 4, 3, 4) \), then the largest composite rank decreases to 1.

\[ \]
To state the full result we introduce what will be the universal sequences $\lambda^{(b)}$ of lengths: Set $\lambda^{(1)} = (1)$, and, inductively, take $\lambda^{(b)}$ to be the result of adding $b$ at the beginning of, at the end of, and in between every pair of elements of, $\lambda^{(b-1)}$. Thus for example

$$
\lambda^{(1)} = (1), \\
\lambda^{(2)} = (2, 1, 2), \\
\lambda^{(3)} = (3, 2, 3, 1, 3, 2, 3), \\
\lambda^{(4)} = (4, 3, 4, 2, 4, 3, 4, 1, 4, 3, 4, 2, 4, 3, 4).
$$

Note that $\lambda^{(b)}$ has $2^b - 1$ elements.

We say that a sequence $\lambda \in \mathbb{N}^m$ is embeddable in $\mu \in \mathbb{N}^n$ if there is a strictly increasing function $\sigma : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that $\lambda_i = \mu_{\sigma(i)}$ for all $1 \leq i \leq m$ (we write $\sigma : \lambda \to \mu$).

**Theorem 1 (Embedding).** There is a Harada-Sai sequence with length sequence $\lambda$ iff there is an embedding $\sigma : \lambda \to \lambda^{(b)}$ for some $b$.

The classic Harada-Sai Lemma follows from the “only if” statement, since the length of the sequence $\lambda^{(b)}$ is $2^b - 1$. The proof of the embedding theorem given at the beginning of the next section, seems to us simpler than the previously known proof of the Harada-Sai Lemma (Ringel [5]). A further consequence of the “if” statement of the embedding theorem is that in any Harada-Sai sequence of maximal lengths the composite ranks of all sub-sequences are determined:

**Corollary 2.** If $\varepsilon$ as above is a Harada-Sai sequence of length $s = 2^b - 1$ with $\lambda(M_i) \leq b$ for all $i$, then the composite rank of any subsequence $M_{i_1} \to \cdots \to M_{i_t}$ is precisely $\min \{\lambda^{(b)}_j | i_1 \leq j \leq i_t\}$.

From the embedding theorem we also get a sharper Harada-Sai bound, taking into account the structure of the ring:

**Corollary 3.** Let $R$ be an Artinian algebra and let $l = l(R)$ be the maximum, over all simple left $R$-modules $N$, of the minimum of the lengths of the projective cover and the injective hull of $N$. The length $s$ of any Harada-Sai sequence $\varepsilon$ of left modules with lengths $\leq b$ is bounded by $s \leq 2^b - l + 1(2^l - 1) + 1$.

For example, if $R$ is the ring of $n \times n$ lower-triangular matrices, then $l(R) = \lfloor n/2 \rfloor$.

In view of Corollary 2, we define the rank of an embedding $\sigma : \lambda = (\lambda_1 \ldots \lambda_s) \to \lambda^{(b)}$ to be $\text{rank} (\sigma) = \min \{\lambda^{(b)}_j | \sigma(1) \leq j \leq \sigma(s)\}$. The following more precise version of Theorem 1 is the main result of this paper.

**Theorem 4 (Rank).** There is a Harada-Sai sequence with length sequence $\lambda$ and composite rank $r$ iff there is an embedding $\sigma : \lambda \to \lambda^{(b)}$ such that $r = \text{rank} (\sigma)$. 
It is easy to decide whether or not a sequence \( \lambda = (\lambda_1, \ldots, \lambda_s) \) is embeddable with given rank in \( \lambda^{(b)} \) (of course we may take \( b = \max \{ \lambda_i \} \)): We define the “optimal” embedding inductively by
\[
\sigma_{\text{opt}}(i) = \min \{ j \mid j > \sigma_{\text{opt}}(i - 1) \text{ and } \lambda_i = \lambda_j^{(b)} \}
\]
as long as such an index \( j \) exists.

**Proposition 5.** A length sequence \( \lambda = (\lambda_1, \ldots, \lambda_s) \) is embeddable in \( \lambda^{(b)} \), iff \( \sigma_{\text{opt}}(i) \) is defined for all \( i \leq s \). There exists a rank \( r \) embedding of \( \lambda \) in \( \lambda^{(b)} \) iff \( r \leq \text{rank}(\sigma_{\text{opt}}) \).

Here is an open problem: Find a generalization of Theorem 1 with hypothesis depending on the maps \( f_i \) and not the modules \( M_i \) in \( \varepsilon \). For example, one might want to assume that no composition of the maps \( f_i \) is an isomorphism, and no image of such a composition is a direct summand. Does this suffice to bound the length of the sequence?

**Proofs**

**Proof of Theorem 1.** We begin by showing that the length sequence \( \lambda = (\lambda_1, \ldots, \lambda_s) \) of a Harada-Sai sequence \( \varepsilon \) as above is embeddable in \( \lambda^{(b)} \), where \( b = \max \{ \lambda_i \} \). We do this by induction on \( b \), the case \( b = 1 \) being trivial.

If \( \lambda_i = \lambda_{i+1} = b \) then we can interpolate some indecomposable summand of the image of \( M_i \) in \( M_{i+1} \) between \( M_i \) and \( M_{i+1} \) and get a new Harada-Sai sequence one step longer than the original. Repeating this process, we may assume that no two consecutive \( \lambda_i \) are equal to \( b \).

Any composition \( g := f_j f_{j-1} \cdots f_1 : M_i \to M_j \) of consecutive maps in a Harada-Sai sequence is a non-isomorphism. For if \( g \) were an isomorphism then \( \lambda(M_{i+1}) > \lambda(M_i) \) to avoid \( f_i \) being an isomorphism, and thus \( M_i \to M_{i+1} \to M_j \xrightarrow{g^{-1}} M_i \) exhibits \( M_i \) as a proper direct summand of \( M_{i+1} \), contradicting the definition.

Let \( \mu = (\mu_1, \ldots, \mu_t) \) be the subsequence of \( \lambda \) consisting of those terms that are \( < b \). By induction there is an embedding \( \sigma' : \mu \to \lambda^{(b-1)} \). Since every second element of \( \lambda^{(b)} \) is \( b \), and the remaining elements of \( \lambda^{(b)} \) make up \( \lambda^{(b-1)} \), this embedding can be extended to the desired embedding of \( \lambda \) in \( \lambda^{(b)} \), completing the proof of embeddability.

The other implication of Theorem 1 follows from the existence of a maximal Harada-Sai sequence, constructed below. \( \square \)

**Proof of Corollary 2.** It is clear that the given rank is an upper bound. If the actual rank were less, we could form a new Harada-Sai sequence of the form
\[
M_1 \to M_2 \to \cdots \to M_i \to N \to M_i \to M_{i+1} \to M_s
\]
where \( N \) is an indecomposable summand of \( \text{Im} M_i \to M_i \), and the length sequence of this new sequence would not be embeddable. \( \square \)
Proof of Corollary 3. Let \( \varepsilon \) be a Harada-Sai sequence of \( R \)-modules of lengths bounded by \( b \), and consider the projective cover \( P(M_1) \to M_1 \). There is an indecomposable direct summand \( P' \) of \( P(M_1) \) such that the composition \( P' \to M_1 \to M_2 \to \cdots \to M_i \) is not zero. Let \( P \) be the image of \( P' \) in \( M_1 \). Let \( N \) be the simple module which is the top of \( P \). The module \( N \) must occur in the image of the composite map \( P \to M_1 \to M_s \), so we can find a map \( M_s \to I' \), where \( I' = I(N) \) is the injective hull of \( N \), so that the composite \( P \to M_1 \to M_s \to I' \) is nonzero. We let \( I \) be the image of \( M_i \) in \( I' \). Both \( P \) and \( I \) have lengths \( \leq \min(b, l) \). By the embeddability statement of Theorem 1, the sequence \( (\lambda(P), \lambda(M_1), \ldots, \lambda(M_j), \lambda(I)) \) is embeddable in \( \lambda(b) \). But the lengths of the subsequences of \( \lambda(b) \) starting and ending with numbers \( \leq l \) are bounded by \( 2^{b-l+1}(2^{l-1} - 1) + 1 \). \( \square \)

Proof of Proposition 5. The first statement is immediate, and it is easy to see that no embedding has rank \( > \tau_{\text{opt}} := \text{rank}(\sigma_{\text{opt}}) \). If \( 0 < r < r_{\text{opt}} \) then we construct a new embedding \( \tau : \lambda \to \lambda(b) \) such that \( r = \text{rank}(\tau) \). Let \( s, t, u, v \) be the smallest integers such that \( \lambda_s(b) = r_{\text{opt}}, \lambda_t(b) = r_{\text{opt}} - 1, \lambda_u(b) = r, \lambda_v(b) = r - 1 \). We set \( \tau(i) = \sigma(i) \) if \( \sigma(i) \leq t \). There is an embedding \( \gamma \) identifying that part of the sequence \( \lambda(b) \) between \( s \) and \( t \) with that part between \( u \) and \( v \). We define \( \tau(i) = \gamma \sigma(i) \) for \( i \) such that \( \sigma(i) > t \). \( \square \)

Proof of Theorem 4. Let \( \varepsilon \) be a Harada-Sai sequence and let \( \sigma_{\text{opt}} \) be its optimal embedding into \( \lambda(b) \). We now proceed by induction on the length \( s \). Set
\[
m = \min \{ \lambda_i(b) \mid \sigma_{\text{opt}}(1) \leq i \leq \sigma_{\text{opt}}(s) \},
\]
the desired bound. If any \( \lambda_i \) is equal to \( m \) we are done, so we assume \( \lambda_i > m \) for all \( i \). Let \( k_0 \) be the smallest integer \( k \) such that \( \lambda_k(b) = m \). Since the embedding \( \sigma_{\text{opt}} \) is optimal, we have \( \sigma_{\text{opt}}(1) < k_0 < \sigma_{\text{opt}}(s) \). On the other hand, if \( k_0 < \sigma_{\text{opt}}(s - 1) \) we are done by induction, so we may assume that \( \sigma_{\text{opt}}(s - 1) < k_0 < \sigma_{\text{opt}}(s) \).

Suppose that for some \( i < j \) we have \( \lambda_i = \lambda_j = m + 1 \). As in the proof of Theorem 1 we see that the rank of the composite \( M_i \to M_j \) and with it the composite rank of \( \varepsilon \), is \( \leq m \) as required. Thus we may further assume that at most one \( \lambda_i \) is equal to \( m + 1 \).

The cases \( s \leq 2 \) are trivial. Assuming that \( s \geq 3 \) we will show that for some integer \( 1 \leq t < s \) the compositions \( g : M_1 \to M_t \) and \( h : M_t \to M_s \) both have rank \( \leq m + 1 \), while \( M_t \) has length \( > m + 1 \). It follows that the rank of the composite \( hg : M_1 \to M_s \) has rank at most \( m + 1 \); and if its rank is equal to \( m + 1 \), then as above the maps
\[
\text{Im } g \to M_t \to \text{Im } hg \cong \text{Im } g
\]
show that the image of \( g \) is a proper summand of \( M_t \), contradicting the definition. Thus it suffices for the rank statement to produce a \( t \) with the properties above.

Since every number greater than \( m \) comes before \( m \) in the sequence \( \lambda(b) \) there is a number \( k < k_0 \) with \( \sigma_{\text{opt}}(1) \leq k \leq \sigma_{\text{opt}}(s) \) and \( \lambda_k(b) = m + 1 \). Let \( k_1 \) be the minimal such number, and choose \( t' \) minimal such that \( \sigma_{\text{opt}}(t') \geq k_1 \). If \( \lambda_{t'} > m + 1 \) then we choose \( t = t' \), while if \( \lambda_{t'} = m + 1 \) then we choose \( t = t' + 1 \). We check the properties of \( t \) as follows, using
follows that every number between \( b \) and \( \lambda_i^{(b)} \) appears before \( \lambda_i^{(b)} \) in the sequence \( \lambda^{(b)} \):

- \( t > 1 \): The case \( t = 1 \) would require \( k_1 < \sigma_{\text{opt}}(1) = \sigma_{\text{opt}}(t) = \sigma_{\text{opt}}(t') < k_0 \). As the subsequence of \( \lambda^{(b)} \) between \( m + 1 = \lambda_{k_1}^{(b)} \) and \( m = \lambda_{k_0}^{(b)} \) is a repeat of the part of the sequence before \( \lambda_{k_1}^{(b)} \), this contradicts the optimality of \( \sigma_{\text{opt}} \).

- \( t < s \): If \( \sigma_{\text{opt}}(s - 1) < k_1 \) then since every number \( \geq m + 1 \) occurs between \( \lambda_{k_1}^{(b)} \) and \( \lambda_{k_0}^{(b)} \), we would have \( \sigma_{\text{opt}}(s) < k_0 \), a contradiction. If actually \( \sigma_{\text{opt}}(s - 1) = k_1 \), so that \( \lambda_{s-1} = m + 1 \), then the same contradiction would occur unless \( \lambda_s = m + 1 \), contradicting our assumption that at most one of the \( \lambda_k \) is equal to \( m + 1 \). Thus \( \sigma_{\text{opt}}(s - 1) > k_1 \). It follows that \( t \leq s - 1 \) unless \( t' = s - 1 \), \( \lambda_{s-1} = m + 1 \). This last case again contradicts the optimality of \( \sigma_{\text{opt}} \), since \( \lambda_s \) could be found strictly between \( \lambda_{t'}^{(b)} = m + 1 \) and \( \lambda_{k_0}^{(b)} = m \).

- rank \((g : M_1 \to M_s) \leq m + 1\): This follows from the induction, since

\[ \sigma_{\text{opt}}(1) \leq k_1 \leq \sigma_{\text{opt}}(t). \]

- rank \((h : M_s \to M_s) \leq m + 1\): The sequence \((\lambda_1^{(b)}, \ldots, \lambda_{s-1}^{(b)})\) is the same as the sequence \((\lambda_{k_1+1}^{(b)}, \ldots, \lambda_{k_0-1}^{(b)})\); it follows that if \( \tau \) is the optimal embedding of the length sequence \((\mu_1 = \lambda_1, \ldots, \mu_{s-1} = \lambda_s)\) then \( \tau(1) \leq k_1 \leq \tau(s - t + 1) \). The desired conclusion follows by our induction.

- length \((M_s) > m + 1\): In the sequence \( \lambda^{(b)} \) there is an occurrence of \( m \) between the first two occurrences of \( m + 1 \). The first occurrence of \( m + 1 \) is in the \( k_1 \) place, that of \( m \) is in the \( k_0 \) place. Since \( k_1 < \sigma_{\text{opt}}(t) < k_0 \) we thus have \( \sigma_{\text{opt}}(t) = m + 1 \), and it follows that \( \sigma_{\text{opt}}(t) > m + 1 \). This completes the proof of the rank condition.

### Construction of examples

By Corollary 2 the existence statements of Theorems 1 and 4 follow from the existence, for each \( b \), of a Harada-Sai sequence with length sequence \( \lambda^{(b)} \). By Corollary 3 there is no ring of finite length over which all such possibilities exist. One of the simplest rings over which they could all exist is the ring \( R = K[x,y]/(xy) \), where \( K \) is a field and \( K[x,y] \) denotes the commutative polynomial ring; we shall construct them there. (Harada-Sai sequences of maximum length have been constructed over certain noncommutative rings of finite representation type by Bongartz [1].)

For any (noncommutative) word \( w \) in \( x \) and \( y \) we construct an indecomposable \( R \)-module \( M(w) \) of length one more than the number of letters in the word. Each module is constructed with a particular basis, and we will only consider maps that take basis elements to basis elements. Among the basis elements, one is distinguished; we call it \( \star \).

Rather than give a formal definition we describe the module \( M(w) \) for the word \( w = yx^2y^2x^3 \): In this case \( M(w) \) has a \( K \)-basis of \( 8 = (1 + 2 + 2 + 3) \) elements labelled \( \bullet \) and one element labelled \( \star \), with multiplication table given by
Note that $y \star = 0$ and $\star \notin xM(w)$; equivalently, $\star$ appears at the right hand side of the diagram above.

To construct the Harada-Sai sequences we use three functors:

$\diamond D = \text{Hom}_K(-, K) : \text{mod}_R \to \text{mod}_R$ is the duality functor; the distinguished basis vector $\star$ of $\text{Hom}_K(M, K)$ is taken to be the dual basis vector to the distinguished basis vector $\star$ of $M$.

$\diamond E : \text{mod}_R \to \text{mod}_R$ the induced by the ring automorphism $R \to R, x \mapsto y, y \mapsto x$.

Observe that $EM(w) = M(\hat{w})$, where $\hat{w}$ is obtained from $w$ by exchanging $x$ and $y$. If $f : M(w_1) \to M(w_2)$ satisfies $f(\star) = \star$, then $E(f) : M(\hat{w}_2) \to M(\hat{w}_{1})$ satisfies $E(f)(\star) = \star$.

$\diamond F(M(w)) := M(wx)$. Note that $M(w)$ is naturally a submodule of $M(wx)$; we take the new $\star$ to be the newly added basis vector. If $f : M(v) \to M(w)$ is a map preserving $\star$, then we take $F(f) : M(wx) \to M(wx)$ to be $f$ extended in such a way as to preserve the new $\star$.

For each $b \in \mathbb{N}$, we construct a Harada-Sai sequence $\varepsilon^{(b)}$ in $\text{mod}_R$, having length sequence $\lambda^{(b)}$. We proceed by induction on $b$, taking $\varepsilon^{(1)}$ to be the sequence whose only terms is the one-dimensional module $M(1)$, where 1 represents the empty word.

Suppose that $\varepsilon^{(b)}$ has the form

$$
\varepsilon^{(b)} : M(w_1) \longrightarrow M(w_2) \longrightarrow \cdots \longrightarrow M(w_{2b-1}) \longrightarrow \cdots \longrightarrow M(w_{2b-1})
$$

and all maps take $\star$ to $\star$. Note that $\lambda^{(b)}_{2b-1} = 1$, so $w_{2b-1} = 1$. We define $\varepsilon^{(b+1)}$ to be the sequence with left segment

$$
\alpha : F(M(w_1)) \longrightarrow F(M(w_2)) \longrightarrow \cdots
$$

$$
\longrightarrow F(M(w_{2b-1})) = M(x) = FE(M(w_{2b-1})) \longrightarrow \cdots \longrightarrow FE(M(w_{2b-1})).
$$

In all these modules $\star$ is outside the radical. Thus we may continue the sequence $\alpha$ with the map $f : FE(M(w_{2b-1})) \to M(1)$ sending $\star$ to $\star$ and sending all the other basis vectors to 0. We define the right-hand segment of $\varepsilon^{(b+1)}$ to be $D(\alpha)$. The whole sequence $\varepsilon^{(b+1)}$
consists of these two together with the module $M(1)$ in the middle. We may represent this symbolically as

\[ \xi^{(b+1)} : \alpha \xrightarrow{f} M(1) \xrightarrow{D(f)} D(\alpha) . \]

Since the composition of all the maps in the sequence sends ★ to ★ it is nonzero, and we are done.  \(\square\)

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Eingegangen 12. Mai 1997