Background Lecture: Flatness and the local criterion for flatness.

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First briefly discuss the basic definitions and properties of flat modules and flat morphisms of schemes (as in Hartshorne, Chapter III, section 9). In particular, state Theorem 9.9. Then go on to discuss the following version of the local criterion for flatness (this version and a proof can be found for example in Chapter V, Section 3 of Altman and Kleiman, Introduction to Grothendieck Duality Theory, Springer Lecture Notes in Math 146).

Let A be a ring and $I \subset A$ a nilpotent ideal. For an A-module M let $\mathrm{gr}_I^s(M)$ denote $I^sM/I^{s+1}M$, and let $\mathrm{gr}_I^*(M)$ denote $\oplus_s \mathrm{gr}_I^s(M)$. Observe that $\mathrm{gr}_I^*(M)$ is a module over $\mathrm{gr}_I^*(A) = \oplus_s I^s/I^{s+1}$.

Theorem 1. *The following statements are equivalent:*

(i) M *is a flat* A*-module;*

(ii) $M \otimes_A (A/I)$ *is a flat* A/I*-module and* $\mathrm{Tor}_1^A(M, A/I) = 0$*;*

(iii) $\mathrm{Tor}_1^A(M, N) = 0$ *for all* A*-modules* N *annihilated by* I*;*

(iv) $M \otimes_A (A/I)$ *is a flat* (A/I)*-module and the natural map*

$$\mathrm{gr}_J^0(M) \otimes_{A/I} \mathrm{gr}_I^*(A) \to \mathrm{gr}_I^*(M)$$

is an isomorphism.