Background Lecture: Grothendieck Existence Theorem for coherent sheaves.

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The topic of this lecture is the Grothendieck Existence theorem (EGA, Chapter III, Theorem 5.1.4). The statement there is rather technical so I propose that in your presentation you try to keep it as down to earth as possible. To this end, I suggest presenting the above theorem without using formal schemes, and assuming that $f : X \to Y$ is proper and that A is complete local. The statement then becomes the following.

Let $\mathfrak{m} \subset A$ be the maximal ideal and for an integer n let A_n denote $A/\mathfrak{m}^{n+1}$, and let X_n denote $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_n)$. Let $\varprojlim \mathrm{Coh}(X_n)$ denote the category of data $(F_n, \iota_n)_{n\geq 0}$, where F_n is a coherent sheaf on X_n and $\iota_n : F_{n+1}|_{X_n} \to F_n$ is an isomorphism of coherent sheaves on X_n.

The most commonly used version of the Grothendieck Existence Theorem is then the following:

Theorem 1. *The reduction functor*

$$(\textit{coherent sheaves on } X) \to \varprojlim \mathrm{Coh}(X_n)$$

is an equivalence of categories.

In the context of the workshop, an important consequence of this theorem is the following (please try to fit this in if possible):

Theorem 2. *Let A, A_n, etc. be as above. Suppose given for every n a closed subscheme $Z_n \subset \mathbb{P}^r_{A_n}$ (for some fixed r) flat over $\mathrm{Spec}(A_n)$ such that $Z_{n+1} \times_{\mathrm{Spec}(A_{n+1})} \mathrm{Spec}(A_n) \subset \mathbb{P}^r_{A_n}$ is equal to Z_n for all $n \geq 0$. Then there exists a unique closed subscheme $Z \subset \mathbb{P}^r_A$ inducing the Z_n.*

This theorem is obtained by applying the preceding theorem to the ideal sheaves defining the Z_n.