Background lecture: Etaleness

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First, briefly discuss the statements on etaleness contained in Hartshorne's Exercise 10.3 of Chapter III. For intuition, state that etaleness is the analogue in algebraic geometry of a map between complex manifolds being a local isomorphism/biholomorphism. Then focus on the formal criterion for etaleness. This is given as Proposition 6 of Section 2.2 of the book *Neron Models*, by Bosch, Lutkebohmert, and Raynaud. Note that the definitions of smoothness and etalenss given there is different from Hartshorne's; see also Proposition 8 of Section 2.4 for the equivalence of the definitions of smoothness, and the equivalence for etaleness then follows from Proposition 11 of Section 2.2.

Say something about the proofs of the above statements as you have time for, but be sure to leave time to state the following important result:

Theorem 1. *Let $f : X \to Y$ be a morphism locally of finite type, with Y locally Noetherian, and $x \in X$. Then f is etale at x if and only if for every diagram*

$$\begin{array}{ccc} Z = \operatorname{Spec} A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ Z' = \operatorname{Spec} A' & \longrightarrow & Y, \end{array}$$

with $A' \to A$ a surjection, A' a local Artin ring, and Z having image x, we have that the dashed arrow exists and is unique.

This is Proposition 17.14.2 of part 4 of Chapter IV of *EGA*. Note that this differs from the formal criterion for etaleness only in terms of allowing us to examine etaleness at a given point, and more crucially, to restrict A' to be a local Artin ring.

Also note that etaleness at $x \in X$ is defined in terms of the behavior of f on an entire Zariski open neighborhood of x, and this theorem is saying in essence that in the locally Noetherian case, not only can one restrict attention to the local rings (which is true in general), but it is enough to look at their completions.

If you have a few minutes, include the statement of the above-mentioned Proposition 11 of Section 2.2 of *Neron Models*, that smooth is equivalent to etale over affine space, and also Proposition 14, that smooth morphisms have etale sections (feel free to word this result less confusingly).