Background lecture: Smoothness

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First, briefly discuss Hartshorne's definition of smoothness, and the relationship to being nonsingular, and then focus on the formal criterion for smoothness. This is given as Proposition 6 of Section 2.2 of the book *Neron Models*, by Bosch, Lutkebohmert, and Raynaud. Note that the definition of smoothness given there is different from Hartshorne's; see also Proposition 8 of Section 2.4.

Say something about the proofs of the above statements as you have time for, but be sure to leave time to state the following important result:

Theorem 1. *Let $f : X \to Y$ be a morphism locally of finite type, with Y locally Noetherian, and $x \in X$. Then f is smooth at x if and only if for every diagram*

$$\begin{array}{ccc} Z = \operatorname{Spec} A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ Z' = \operatorname{Spec} A' & \longrightarrow & Y, \end{array}$$

with $A' \to A$ a surjection, A' a local Artin ring, and Z having image x, we have that the dashed arrow exists.

This is Proposition 17.14.2 of part 4 of Chapter IV of *EGA*. Note that this differs from the formal criterion for smoothness only in terms of allowing us to examine smoothness at a given point, and more crucially, to restrict A' to be a local Artin ring.

Also note that smoothness at $x \in X$ is defined in terms of the behavior of f on an entire Zariski open neighborhood of x, and this theorem is saying in essence that in the locally Noetherian case, not only can one restrict attention to the local rings (which is true in general), but it is enough to look at their completions.