EXERCISES 1

Exercise 1.1. Let $X = (|X|, \mathscr{O}_X)$ be a scheme and I is a quasi-coherent $\mathscr{O}_X$-module. Show that the ringed space $X[I] := (|X|, \mathscr{O}_X[I])$ is also a scheme.

Exercise 1.2. Let $\mathcal{C}$ be a category and $h : \mathcal{C} \to \mathrm{Func}(\mathcal{C}^\circ, \underline{\mathrm{Set}})$ the Yoneda embedding. Show that for any arrows $X \to Y$ and $Z \to Y$ in $\mathcal{C}$, there is a natural isomorphism $h_{X \times_Z Y} \xrightarrow{\sim} h_X \times_{h_Y} h_Z$.

Exercise 1.3. Let $A \to R$ be a ring homomorphism. Verify that under the identification of $\mathrm{Der}_A(R, \Omega_{R/A})$ with the sections of the diagonal map $R \otimes_A R/J^2 \to R$ given in lecture, the universal derivation $d : R \to \Omega^1_{R/A}$ corresponds to the section given by sending $x \in R$ to $1 \otimes x$.

Exercise 1.4. Let $\mathrm{Aff}_{\mathbb{Z}}$ be the category of affine schemes. The Yoneda functor h gives rise to a related functor

$$h^{\mathrm{Aff}} : \mathrm{Sch}_{\mathbb{Z}} \to \mathrm{Func}(\mathrm{Aff}^\circ_{\mathbb{Z}}, \underline{\mathrm{Set}})$$

which only considers the functor of points for affine schemes. Is this functor still fully faithful?

Cultural note: since $\mathrm{Aff}^\circ_{\mathbb{Z}}$ is equivalent to the category of rings ($\mathbb{Z}$-algebras!), we can also view h as defining a (covariant) functor on the category of rings. When X is an affine scheme of the form $\mathbb{Z}[\{x_i\}]/(\{f_j\}]$, the value of h^{Aff}_X on A is just the set of solutions of the f_j with coordinates in A.

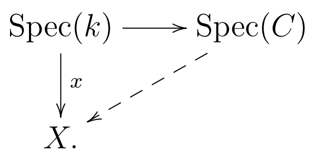
Exercise 1.5. Let R be a ring and $H : \mathrm{Mod}_R \to \mathrm{Set}$ a functor which commutes with finite products. Verify the claim in lecture that $H(I)$ has an R-module structure.

Exercise 1.6. Given a category $\mathcal{C}$ and a functor $F : \mathcal{C}^\circ \to \underline{\mathrm{Set}}$, produce a natural bijection $\hom(h_X, F) \xrightarrow{\sim} F(X)$ for every object X of $\mathcal{C}$.

Exercise 1.7. Let k be a field, X/k a scheme, and $x \in X(k)$ a k-valued point. Let $R = k$ and let $k \to R$ be the identity morphism. Define

$$F : k - \mathrm{Alg}/k \to \mathrm{Set}$$

to the functor sending a diagram $k \to C \to k$ to the set of dotted arrows over $\mathrm{Spec}(k)$ filling in the following diagram:



Show that the hypotheses discussed in lecture for F to have a tangent space hold, and that the resulting k-vector space T_F is canonically isomorphic to the tangent space of X at x.

Suppose now that $x \in X$ is an arbitrary point, with residue field $k(x)$. We can define a similar functor on $k - \mathrm{Alg}/k(x)$ by trying to fill in diagrams

$$\begin{array}{ccc} \mathrm{Spec}(k(x)) & \longrightarrow & \mathrm{Spec}(C) \\ \downarrow{\scriptstyle x} & \swarrow & \\ X. & & \end{array}$$

Do we still get the tangent space to X at x? (What happens when x is a generic point of X?) How does this construction relate to the sheaf of relative differentials $\Omega^1_{X/k}$?

Exercise 1.8. *A criterion for representability.* (Save this for last. If you find it impenetrable, read it a few times and come back to it tomorrow.)

Let $\mathcal{C}$ be a category. A *subfunctor* of a functor F is an equivalence class of pairs consisting of a functor G and a natural transformation $G \to F$ such that for all $T \in \mathcal{C}$, the induced map $G(T) \to F(T)$ is injective. (The equivalence relation is given by isomorphisms of G and G' which commute with the inclusions in F.) If $G \to F$ is a subfunctor and $H \to F$ is arbitrary, check that $G \times_F H \to H$ is a subfunctor.

Suppose S is a fixed base scheme. Write Sch_S for the category of S-schemes.

Definition 1.9. A subfunctor $G \hookrightarrow F$ on Sch_S is an *open subfunctor* if for every S-scheme X and every map $h_X \to F$, the induced subfunctor $G \times_F h_X \hookrightarrow h_X$ has the property that $G \times_F h_X$ is representable, say it is isomorphic to h_Y, and the map $h_Y \to h_X$ arises from an open immersion $Y \hookrightarrow X$.

A collection of open subfunctors $G_i \to F$ is an *open covering* if for every X and every map $h_X \to F$ as above, the collection $G_i \times_F h_X \to h_X$ comes from a collection of open immersions $Y_i \hookrightarrow X$ which form a covering in the Zariski topology.

Definition 1.10. A functor $F : \mathrm{Sch}_S^\circ \to \underline{\mathrm{Set}}$ is a *Zariski sheaf* (the terminology will make sense tomorrow) if for all S-schemes X, the association $U \mapsto F(U)$ (for open subschemes $U \subset X$) defines sheaf on X in the Zariski topology.

Prove that a functor $F : \mathrm{Sch}_{\mathbb{Z}}^\circ \to \underline{\mathrm{Set}}$ is representable if and only if it is a Zariski sheaf and has an open cover by representable subsheaves.