EXERCISES 2

Exercise 2.1. Given yesterday's description of the functor of points of $\mathbb{P}^n$, what are the subfunctors corresponding to the standard open cover by copies of $\mathbb{A}^n$?

Exercise 2.2. Let $j : X_0 \hookrightarrow X$ be a closed immersion of schemes defined by a nilpotent ideal. If X_0 is affine, then X is also affine.

Exercise 2.3. Let $X \to S = \operatorname{Spec}(R)$ be a smooth morphism and let I be an R-module. Prove that even when X is not separated, then there is a canonical bijection

$$\operatorname{Def}_X(R[I]) \to H^1(X, T_X \otimes I).$$

Exercise 2.4. Prove that the R-module structure on T_{Def_X} defined in lecture agrees with the standard R-module structure on $H^1(X, T_X)$ under the identification in exercise 2.3.

Exercise 2.5. Show that F is prorepresentable if and only if there exists a complete Noetherian local ring R and a collection $\eta_n \in F(R/\mathfrak{m}^n)$ of elements, compatible with restriction, such that for every $A \in \operatorname{Art}(\Lambda, k)$, and every $\eta \in F(A)$, there exists a unique map $f_\eta : R \to A$ such that if f_η factors through $R/\mathfrak{m}^n$, we have that η is the image of η_n.

Exercise 2.6. (Co)tangent spaces of rings in $\widehat{\operatorname{Art}}(\Lambda, k)$. Given $R \in \widehat{\operatorname{Art}}(\Lambda, k)$, we define the **cotangent space** of R (relative to Λ), by

$$T_R^* = \mathfrak{m}/(\mathfrak{m}^2 + \mathfrak{n}R),$$

where $\mathfrak{m}$ is the maximal ideal of R, and $\mathfrak{n}$ is the maximal ideal of Λ.

(1) Show that $T_R^* = (T_{h_R})^*$.
(2) Show that if $R \to R'$ in $\widehat{\operatorname{Art}}(\Lambda, k)$ induces a surjection $T_R^* \twoheadrightarrow T_{R'}^*$, then $R \to R'$ is surjective. [Note that compared to Lemma II.7.4 of Hartshorne, we restrict our attention to complete rings, but don't need any finiteness hypotheses.]

Exercise 2.7. We'll show that any two hulls are isomorphic (although not canonically isomorphic, in general).

(1) Suppose that $f : A \to A$ is a surjective endomorphism of Noetherian rings. Then f is an isomorphism.
(2) If $F \to G$ is a smooth morphism of predeformation functors, then $\hat{F} \to \hat{G}$ is surjective, in the sense that for every $R \in \widehat{\operatorname{Art}}(\Lambda, k)$, we have

$$\hat{F}(R) \twoheadrightarrow \hat{G}(R).$$

(3) Use (1) and (2) and exercise 2.6 to prove that any two hulls for a predeformation functor F are isomorphic, in the sense that if (R, ξ) and (R', ξ') are hulls for f, there is an isomorphism $R \xrightarrow{\sim} R'$ sending ξ to ξ'.

Exercise 2.8. Let S be a scheme and $\mathscr{F}$ a quasi-coherent sheaf on S. Define a functor

$$\widetilde{\mathscr{F}} : \operatorname{Sch}_S^\circ \to \underline{\operatorname{Set}}$$

by sending $f : T \to S$ to $\Gamma(T, f^*\mathscr{F})$.

(1) If $\mathscr{F} = \mathscr{O}_S$, then note (show?) that $\widetilde{\mathscr{F}} \cong h_{\mathbb{A}^1}$. Thus, as we showed in today's first lecture, we know that $\widetilde{\mathscr{O}_S}$ is an fppf sheaf.
(2) Show that $\widetilde{\mathscr{F}}$ is an fppf sheaf for any quasi-coherent $\mathscr{F}$ as follows: it suffices (by the lemma from class) to show the sheaf condition relative to a faithfully flat morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ of affine schemes (over S). Writing $\mathscr{F}|_{\operatorname{Spec} A}$ as the sheaf associated to an A-module M, show that the sheaf condition is equivalent to the exactness of the diagram

$$A \otimes_A M \longrightarrow B \otimes_A M \rightrightarrows B \otimes_A B \otimes_A M,$$

where the maps come from tensoring the existing maps with M.
(3) Show that the diagram above is exact. (Hint: use the same proof as in the lecture – reduce to the case where there is an augmentation $B \to A$.)

Exercise 2.9. Consider the sheaf $\widetilde{\mathscr{O}_{\mathbb{A}^1}}$ from exercise 2.8. Choosing a coordinate on $\mathbb{A}^1$ yields an element $t \in \widetilde{\mathscr{O}_{\mathbb{A}^1}}(\mathbb{A}^1)$.

(1) What is the kernel of the map $\widetilde{\mathscr{O}_{\mathbb{A}^1}} \to \widetilde{\mathscr{O}_{\mathbb{A}^1}}$ given by multiplication by t?
(2) What if we restrict to the small étale site of $\mathbb{A}^1$?

Exercise 2.10. Let X be an irreducible scheme. Show that the (Zariski) topological space underlying X is simply connected. What happens if X is reducible?