EXERCISES 3

Exercise 3.1. Let $f : X' \to X$ be a faithfully flat quasi-compact (fpqc) morphism. Using the fact that f is an effective descent morphism for quasi-coherent sheaves, show that it is an effective descent morphism for relatively affine X-schemes (i.e., for schemes of the form $\mathbf{Spec}\,\mathscr{A}$ where $\mathscr{A}$ is a quasi-coherent sheaf of algebras over the structure sheaf).

Exercise 3.2. Let $f : X \to S$ be a smooth morphism of schemes with $S = \mathrm{Spec}(A)$ affine.

(a) Define a map of sheaves

$$d\log : \mathscr{O}_X^* \to \Omega^1_{X/S}$$

by sending a local section $f \in \mathcal{O}_X^*$ to $f^{-1}df$. Show that $d\log$ is a homomorphism. Let

$$c_1 : \mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*) \to H^1(X, \Omega^1_{X/S})$$

be the map obtained by passing to cohomology.

(b) Let L be a line bundle on X, and let $f' : X' \to S[\epsilon]$ be a smooth lifting of X to $S[\epsilon]$. Recall that this lifting is defined by a class $c(f') \in H^1(X, T_{X/S})$. Now consider the pairing

$$\langle \cdot, \cdot \rangle : H^1(X, T_{X/S}) \times H^1(X, \Omega^1_{X/S}) \to H^2(X, \mathscr{O}_X)$$

induced by the natural pairing

$$T_{X/S} \otimes \Omega^1_{X/S} \to \mathscr{O}_X.$$

Show that the obstruction $o(L) \in H^2(X, \mathcal{O}_X)$ to lifting L to a line bundle on X' is equal to $\langle c_1(L), c(f') \rangle$.

(c) This can be generalized as follows. Consider a deformation situation $A' \to A \to A_0$ with $A_0 = k$ a field, and let $(X, L)/A$ be a smooth A-scheme with an ample line bundle L, and let (X_0, L_0) be the reduction to k. Assume that there exists a smooth lifting X'/A' of A and that the map

$$\langle -, c_1(L_0) \rangle : H^1(X_0, T_{X_0/k}) \to H^2(X_0, \mathscr{O}_{X_0})$$

is surjective. Then the pair (X, L) lifts to A'.

Exercise 3.3. We can set up descent theory for schemes in parallel with what we did for quasi-coherent sheaves. (We will go into more detail soon.) In this exercise, we will study various conditions under which descent is effective.

Suppose $f : X' \to X$ is an fpqc morphism. Write $X'' := X' \times_X X'$ and p_1 and p_2 for the two projections. Given an X' scheme Y, let p_i^*Y denote the base change of Y via the projection p_i.

(a) Let $Y \to X'$ be a separated X'-scheme along with a descent datum $\phi : p_1^*Y \xrightarrow{\sim} p_2^*Y$. Say that an open subscheme $U \subset Y$ is *stable under* ϕ if the induced map $\phi|_{p_1^*U}$ gives a descent datum $p_1^*U \xrightarrow{\sim} p_2^*U$. Show that if Y has an open affine cover which is stable under ϕ then Y descends to X.

(b) Now suppose X and X' are affine and assume that $Y \to X'$ is proper and smooth. Using the previous part, show that if the relative canonical sheaf $\omega_{Y/X'}$ is X'-ample then any descent datum on Y is effective.

Exercise 3.4. Let $S = \mathrm{Spec}(R)$ be an affine scheme and $f : A \to S$ an abelian scheme over S. Let $R' \to R$ be a surjection of rings with square-zero kernel I, and let $o(f) \in H^2(A, \mathcal{O}_A \otimes I)$ be the obstruction to lifting A to $S' = \mathrm{Spec}(R')$. The purpose of this exercise is to show that $o(f) = 0$.

(a) Let $g : B \to S$ be the abelian scheme $A \times_S A$. Show that the two maps

$$i_1, i_2 : H^2(A, T_{A/S} \otimes I) \to H^2(B, T_{B/S} \otimes I)$$

induced by the projections $B \rightrightarrows A$ are injective.

(b) Show that the obstruction $o(g) \in H^2(B, T_{B/S} \otimes I)$ to lifting B to S' is given by

$$o(g) = i_1(o(f)) + i_2(o(f)).$$

(c) Define $\alpha : B \to B$ to be the automorphism given on scheme-valued points by

$$(x, y) \mapsto (x + y, y).$$

Show that

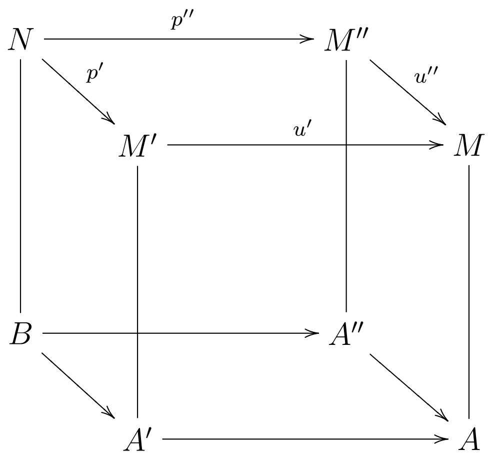
$$o(g) = \alpha^* o(g) = 2i_1(o(f)) + i_2(o(f)).$$

(d) Deduce that $i_1(o(f)) = 0$, and therefore also that $o(f) = 0$.

Exercise 3.5. Let A be a ring, with nilpotent ideal J, and $u : M \to N$ a homomorphism of A-modules, with N flat over A. If $\bar{u} : M/JM \to N/JN$ is an isomorphism, then u is an isomorphism.

Exercise 3.6. Conclude from the previous exercise that M is flat over an Artin local ring A if and only if it is free.

Exercise 3.7. Consider a commutative diagram



of compatible ring and module homomorphisms, where $B = A' \times_A A''$, $N = M' \times_M M''$, and M' and M'' are flat over A' and A'', respectively. Suppose

(i) $A'' \twoheadrightarrow A$, with nilpotent kernel,

(ii) u' induces $M' \otimes_{A'} A \xrightarrow{\sim} M$, and similarly for u''.

(a) Use the local criterion for flatness to show: N is flat over B, and p' induces $N \otimes_B A' \xrightarrow{\sim} M'$, and similarly for p''.

(b) Suppose that with the above notation, we also have a B-module L and maps $q' : L \to M'$ and $q'' : L \to M''$ such that q' induces $L \otimes_B A' \xrightarrow{\sim} M'$. Then the map $q' \times q'' : L \to N$ is an isomorphism.

Exercise 3.8. Let L/K be a finite Galois field extension Galois group G.

(a) Show that there is an isomorphism $L \otimes_K L \cong \prod_{g \in G} L$ such that the two maps $L \to L \otimes_K L$ have the following property: the first is the identity on each factor, while the second maps via the automorphism g to the factor parametrized by $g \in G$. Thus, the statement that

$$K \longrightarrow L \rightrightarrows L \otimes_K L$$

is exact is the same as the statement that the fixed elements of L under G are precisely the elements of K.

(b) Let V be a vector space over L. Show that a descent datum for V is the same as a semilinear G-action: this means that to each $g \in G$ is associated a K-linear map $\sigma_g : V \to V$ such that for any $\alpha \in L$ and $v \in V$, $\sigma_g(\alpha v) = g(\alpha)\sigma_g(v)$, and such that $\sigma_g \circ \sigma_h = \sigma_{gh}$ for all $g, h \in G$. Call an L-vector space with a semilinear G-action a *semilinear G-space*. (Note that any K-vector space W gives rise to a semilinear G-space by letting g act on $W \otimes L$ via its action on the second tensor factor.)

(c) Show that descent theory for quasi-coherent sheaves on $\operatorname{Spec} L \to \operatorname{Spec} K$ is thus equivalent to the following statement: if V is a semilinear G-space, then the natural map $V^G \otimes_K L \to L$ is an isomorphism of semilinear G-spaces. (In other words, every semilinear G-space has a basis consisting of invariant elements.)

Exercise 3.9. Let X be a Noetherian scheme and $f : Y \to X$ a finite étale morphism.

(a) Show that any section σ of f induces a splitting $Y = Y_1 \sqcup Y_2$ such that each $Y_i \to X$ is finite étale and σ is an isomorphism of X onto Y_1. (To do this, it is enough to show that σ is an open immersion and a closed immersion. It is a closed immersion because f is finite and hence separated. To prove that it is an open immersion, note that $\Omega^1_{Y/X} = 0$, so that the diagonal $\Delta : Y \to Y \times_X Y$ is a closed and open immersion [use the fact that the ideal sheaf of the diagonal has the property that it equals its square]. Now show that the section σ is the base change of Δ by the map $(\sigma, \mathrm{id}) : X \times_X Y \to Y \times_X Y$. This is an example of "reduction to the universal section of Y/X" – any section of any base change of Y/X comes by pullback from the diagonal, which is a section of the base change of Y/X along $Y \to X$.)

(b) Use part (a) to conclude that there is an étale covering $U_i \to X$ such that the pullback of Y to U_i is a constant sheaf, i.e., is represented by a disjoint union of copies of U_i. Thus, finite étale maps are "covering spaces in the étale topology" in the sense that they look formally the same as covering spaces in classical topology.