EXERCISES 4

Exercise 4.1. Let $k[t^2, t^3] \to k[t]$ be the normalization of a cusp (corresponding to the normalization map $\mathbb{A}^1 \to \operatorname{Spec} k[x,y]/(y^2 - x^3)$). Show that the residue field of the point $t = 0$ on $\mathbb{A}^1$ cannot have a descent datum. (Note: you cannot directly use descent theory as we developed it for the normalization morphism because this map does not satisfy any of our conditions.)

Exercise 4.2. Let $X \to S$ be a smooth morphism of schemes, and let $S \hookrightarrow S'$ be a nilpotent immersion defined by a square-zero ideal J. Fix a line bundle L on X. Let $P \to S$ denote $X \times_S X$ and let I_Δ denote the ideal of the diagonal of X. Let $\mathscr{D}^1_{X/S}(L)$ denote the sheaf

$$\mathscr{H}om(\mathscr{O}_P/I_\Delta^2 \otimes_{\mathscr{O}_X} L, L).$$

Here the tensor product $\mathscr{O}_P/I_\Delta^2 \otimes_{\mathscr{O}_X} L$ is taken with respect to the right $\mathscr{O}_X$-module structure on $\mathscr{O}_P/I_\Delta^2$ and $\mathscr{O}_P/I_\Delta^2 \otimes_{\mathscr{O}_X} L$ is viewed as an $\mathscr{O}_X$-module via the left $\mathscr{O}_X$-module structure on $\mathscr{O}_P/I_\Delta^2$. Consider the problem of finding a pair $(X', L')/S'$, where X'/S' is a smooth lifting of X and L' is a line bundle lifting L. Show the following:

(a) There is a canonical obstruction $o(X, L) \in H^2(X, \mathscr{D}^1_{X/S}(L) \otimes J)$ whose vanishing is necessary and sufficient for the existence of a lifting (X', L') of (X, L).

(b) If $o(X, L) = 0$ then the set of isomorphism classes of liftings (X', L') form a torsor under $H^1(X, \mathscr{D}^1_{X/S}(L) \otimes J)$.

(c) For any lifting (X', L') the group of automorphisms is canonically isomorphic to

$$H^0(X, \mathscr{D}^1_{X/S}(L) \otimes J).$$

Exercise 4.3. Consider the following variant of exercise 4.2. With notation as above, assume in addition given a section $s \in H^0(X, L)$. Consider the problem of finding a triple (X', L', s'), where (X', L') is a lifting of (X, L) to S' and $s' \in H^0(X', L')$ is a lifting of s.

The section s defines a map

$$\delta : \mathscr{D}^1_{X/S}(L) = \mathscr{H}om(\mathscr{O}_P/I_\Delta^2 \otimes_{\mathscr{O}_X} L, L) \to L, \quad \partial \mapsto \partial(1 \otimes s),$$

so we have a complex

$$\mathscr{D}^1_{X/S}(L) \otimes J \xrightarrow{\delta} L \otimes J.$$

Show the following:

(a) There is a canonical obstruction $o(X, L, s) \in H^2(X, \mathscr{D}^1_{X/S}(L) \otimes J \to L \otimes J)$ whose vanishing is necessary and sufficient for the existence of a lifting (X', L', s') of (X, L, s).

(b) If $o(X, L, s) = 0$ then the set of isomorphism classes of liftings (X', L', s') form a torsor under $H^1(X, \mathscr{D}^1_{X/S}(L) \otimes J \to L \otimes J)$.

(c) For any lifting (X', L', s') the group of automorphisms is canonically isomorphic to $H^0(X, \mathscr{D}^1_{X/S}(L) \otimes J \to L \otimes J)$.

Exercise 4.4. Convince yourself that the proof that Def_X is a deformation functor also shows that $\mathrm{Def}_{\mathcal{E}}$ is a deformation functor, if we are given X_Λ and $\mathcal{E}$ on X. What are the differences in the argument for the two cases?

Exercise 4.5. Let $X' \to X$ be a faithfully flat morphism of T-schemes and $\mathscr{F}$ a quasi-coherent sheaf. Show that $\mathscr{F}$ is

(1) T-flat;
(2) of finite type;
(3) of finite presentation;
(4) locally free;
(5) try other conditions!

if and only if $f^*\mathscr{F}$ has the corresponding property as a quasi-coherent sheaf on X'. (First think about how to do this for rings and modules, and then note that the exercise is local in the Zariski topology on both X and X'.)

Exercise 4.6. Let F be a deformation functor, and $A' \to A$ a small thickening with kernel I.

(a) For every $\eta \in F(A)$, the set of $\eta' \in F(A')$ restricting to η, when non-empty, has a transitive action of $T_F \otimes_k I$. This action commutes with morphisms of deformation functors $F' \to F$. Hint: use the isomorphism

$$A' \times_A A' \xrightarrow{\sim} A' \times_k k[I]$$

induces by $(x, y) \mapsto (x, x_0 + y - x)$, where x_0 is the image in k of x.

(b) The condition (H4) is equivalent to the condition that for all small thickenings and all $\eta \in F(A)$ lifting to A', the above action is free.

Exercise 4.7. Let $p : A' \to A$ be a surjection in $\mathrm{Art}(\Lambda, k)$.

(a) p is essential if and only if the induced map $T^*_{A'} \to T^*_A$ is an isomorphism.

(b) Suppose p is a small thickening. Then p is not essential if and only if p has a section, which is to say there exists a homomorphism $s : A \to A'$ such that $p \circ s = 1$.

Exercise 4.8. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{D} \to \mathcal{C}$ a functor. An arrow $\alpha : d_1 \to d_2$ in $\mathcal{D}$ is *Cartesian* with respect to F if the following universal property holds: for any arrow $\beta : d_3 \to d_2$ such that $F(\beta) = F(\alpha)$, there exists a unique $\gamma : d_3 \to d_1$ with $F(\gamma) = \mathrm{id}_{c_1}$ and such that $\alpha \circ \gamma = \beta$.

(a) Let $\mathcal{A}$ be the category whose objects are morphisms $X \to Y$ of schemes and whose arrows are commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{g} & Y'. \end{array}$$

There is a natural functor $\mathcal{D} \to \mathrm{Sch}_{\mathbb{Z}}$. Show that the Cartesian arrows in $\mathcal{D}$ are precisely the Cartesian diagrams, i.e., those which identify X with $X' \times'_Y Y$.

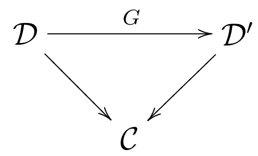
(b) Define a category $\mathrm{QCoh}_{\mathrm{Sch}_{\mathbb{Z}}}$ as follows: objects are pairs $(X, \mathscr{F})$ with X a scheme and $\mathscr{F}$ is a quasi-coherent sheaf on X, and an arrow $(X, \mathscr{F}) \to (X', \mathscr{F}')$ is a pair (f, ϕ) with

$f : X \to X'$ a morphism of schemes and $\phi : \mathscr{F}' \to f_*\mathscr{F}$ a map of quasi-coherent sheaves on X'. Show that the Cartesian arrows are precisely those pairs (f, ϕ) such that the map $f^*\mathscr{F}' \to \mathscr{F}$ induced by the adjunction of f_* and f^* is an isomorphism.

Exercise 4.9. A functor $F : \mathcal{D} \to \mathcal{C}$ between two categories is a *fibered category* if for every arrow $\beta : c_1 \to c_2$ in $\mathcal{C}$ and every d_2 such that $F(d_2) = c_2$, there exists a Cartesian arrow $\alpha : d_1 \to d_2$ such that $F(\alpha) = \beta$.

Show that the examples in the previous exercise are fibered categories.

Exercise 4.10. A morphism of fibered categories over $\mathcal{C}$ is a diagram

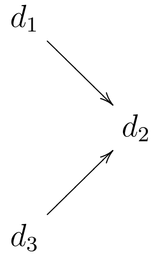


such that G takes Cartesian morphisms to Cartesian morphisms.

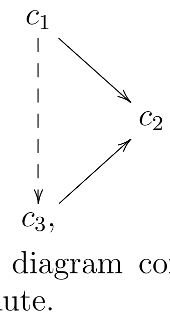
Define a category $\mathrm{QCoh}^p_{\mathcal{A}}$ as follows: objects are pairs $(X \to Y, \mathscr{F})$ with $X \to Y$ an object of $\mathcal{A}$ and $\mathscr{F}$ a Y-flat quasi-coherent sheaf on X; morphisms $(X \to Y, \mathscr{F}) \to (X' \to Y', \mathscr{F}')$ are pairs $((f, g), \phi)$ where (f, g) is a Cartesian arrow in $\mathcal{A}$ and $\phi : \mathscr{F}' \to f_*\mathscr{F}$ is a map of quasi-coherent sheaves on X. Show that the natural map $\mathrm{QCoh}^p_{\mathcal{A}} \to \mathcal{A}$ is a morphism of fibered categories over $\mathrm{Sch}_{\mathbb{Z}}$.

Exercise 4.11. A *category fibered in groupoids* is a fibered category $F : \mathcal{D} \to \mathcal{C}$ such that every arrow in $\mathcal{D}$ is Cartesian.

Show that a functor $F : \mathcal{D} \to \mathcal{C}$ is a category fibered in groupoids if and only if it satisfies the following conditions: first, if $\chi : c \to F(d)$ is an arrow in $\mathcal{C}$ to the image of some object in d, then there is an arrow $d' \to d$ in $\mathcal{D}$ mapping to χ, and second, for any diagram



in $\mathcal{D}$ covering a diagram



any dotted arrow making the bottom diagram commute is the image of a unique arrow $d_1 \to d_3$ making the top diagram commute.

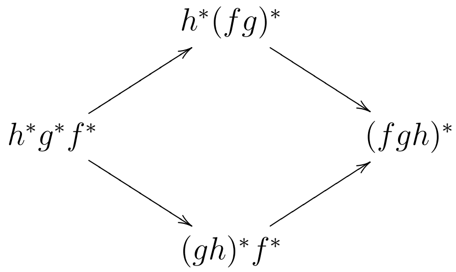
Exercise 4.12. Given a fibered category $F : \mathcal{D} \to \mathcal{C}$ and an object $c \in \mathcal{C}$, the *fiber category* over c, denoted F_c, is the category consisting of objects d with $F(d) = c$ and with $\mathrm{Hom}(d_1, d_2)$

those arrows $\beta : d_1 \to d_2$ with $F(\beta) = \mathrm{id}_c$. Show that F is a category fibered in groupoids if and only if every fiber category is a groupoid. (This explains the name!)

Exercise 4.13. Let $F : \mathcal{D} \to \mathcal{C}$ be a fibered category, and consider the subcategory $\mathcal{D}^{\mathrm{gr}}$ of $\mathcal{D}$ with the same objects and such that $\mathrm{Hom}_{\mathcal{D}^{\mathrm{gr}}}(d_1, d_2)$ is the set of Cartesian arrows with respect to $\mathcal{C}$. Show that the induced functor $\mathcal{D} \to \mathcal{C}$ is a category fibered in groupoids.

Exercise 4.14. Recall that a pseudo-functor G from $\mathcal{C}$ to groupoids consists of:

(1) for each object of c, a groupoid $G(c)$;
(2) for each arrow $\alpha : c_1 \to c_2$ in $\mathcal{C}$ a functor $\alpha^* : G(c_1) \to G(c_2)$;
(3) for each pair of arrows $f : c_1 \to c_2$ and $g : c_2 \to c_3$, an isomorphism of functors $\nu_{fg} : f^* g^* \xrightarrow{\sim} (gf)^*$, such that for any composable triple f, g, h, the resulting diagram



commutes.

Assuming all categories involved are small, show that one can construct a category fibered in groupoids from a pseudo-functor and a pseudo-functor from a category fibered in groupoids.

A morphism $G : \mathcal{D} \to \mathcal{D}'$ of fibered categories over $\mathcal{C}$ is an *equivalence* if the induced functor on fiber categories $G_c : \mathcal{D}_c \to \mathcal{D}'_c$ is an equivalence for all $c \in \mathcal{C}$. Show that if you take a fibered category $\mathcal{D}$, make a pseudo-functor $G_{\mathcal{D}}$, and then make a fibered category $\mathcal{D}_{G_{\mathcal{D}}}$, then there is a natural equivalence $\mathcal{D}_{G_{\mathcal{D}}} \to \mathcal{D}$ of fibered categories.

Exercise 4.15. Is it possible to have two different (i.e., non-equivalent) pseudo-functors which give rise to equivalent categories fibered in groupoids?

Exercise 4.16. Let G be a sheaf of abelian groups on X. Show that $H^1(X, G)$ is naturally in bijection with isomorphism classes of G-torsors.