EXERCISES 6

Exercise 6.1. Keep working on the first five assignments.

Exercise 6.2. Suppose that $X \to B$ is any flat family of curves over a nonsingular curve B, that $0 \in B$ is a geometric point (with residue field k), and that the fiber of the family X_0 has a node at $p \in X_0$. Deduce from the previous exercise that the complete local ring $\hat{O}_{X,p}$ is isomorphic to $k[[x, y, t]]/(xy)$ or $k[[x, y, t]]/(xy - t^k)$ for some integer $k \geq 1$.

Exercise 6.3. We saw that projective space $\mathbb{P}^n$ is the quotient of $\mathbb{A}^{n+1} \setminus \{0\}$ by the natural action of $\mathbf{G}_m$ given by $t(a_1, \ldots, an + 1) \mapsto (ta_1, \ldots, ta_{n+1})$. Given integers $m_1, \ldots, m_{n+1}$ we can also consider the action $t(a_1, \ldots, a_{n+1}) \mapsto (t^{m_1}a_1, \ldots, t^{m_{n+1}}a_{n+1})$. The quotient stack $[(\mathbb{A}^{n+1} \setminus \{0\})/\mathbf{G}_m]$ is denoted $\mathbb{P}(m_1, \ldots, m_{n+1})$. (Thus, $\mathbb{P}(1, \ldots, 1) \cong \mathbb{P}^n$.)

(a) For which tuples is $\mathbb{P}(m_1, \ldots, m_{n+1})$ a sheaf?

(b) For which tuples does $\mathbb{P}(m_1, \ldots, m_{n+1})$ contain an open substack which is a sheaf? (An *open substack* is a map of stacks $\mathscr{X} \to \mathscr{Y}$ which is representable and such that for any $T \to \mathscr{Y}$, the fiber product $\mathscr{X} \times_{\mathscr{Y}} T \to T$ is an open immersion. A stack "is" a sheaf if every fiber category is discrete.)

(c) A *geometric point* of $\mathbb{P}(m_1, \ldots, m_{n+1})$ is an object ξ of the stack over an algebraically closed field. What are the possible stabilizers of geometric points of $\mathbb{P}(m_1, \ldots, m_{n+1})$?

Hint for everything: think about stabilizers of points of $\mathbb{A}^{n+1} \setminus \{0\}$ under the given action of $\mathbf{G}_m$.

Exercise 6.4. Let $K^\cdot, L^\cdot \in \widetilde{C}^{[-1,0]}(T)$ and let $f : K^\cdot \to L^\cdot$ be a morphism. Show that the kernel of the morphism of Picard stacks

$$\mathrm{ch}(f) : \mathrm{ch}(K^\cdot) \to \mathrm{ch}(L^\cdot)$$

is equal to $\mathrm{ch}(\tau_{\leq 0}\mathrm{Cone}(f)[-1])$, where $\mathrm{Cone}(f)$ denotes the cone of the morphism f.