EXERCISES 7

Exercise 7.1. Show that the subscheme of $(\mathbb{A}^1 \setminus \{0,1\}) \times \mathbb{P}^2$ cut out by $Y^2Z = X(X - Z)(X - \lambda Z)$ (where λ is a coordinate on $\mathbb{A}^1$), along with the point $(0 : 1 : 0)$ at infinity, defines an étale surjection $\mathbb{A}^1 \setminus \{0,1\} \to \mathscr{M}_{1,1}$.

Exercise 7.2. (Realization of $\mathscr{M}_{1,1}$ as a quotient stack)

As in lecture, consider the stack $\mathscr{S}$ whose fiber category over T consists of diagrams

$$\begin{array}{ccc} E & \xrightarrow{f} & \mathbb{P}^2_T \\ {\scriptstyle\sigma}\uparrow\downarrow{\scriptstyle\pi} & \swarrow & \\ T, & & \end{array}$$

where π is a proper smooth morphism with geometrically connected fibers of genus 1, σ is a section, and f is a closed immersion such that $f^*\mathscr{O}(1) \cong \mathscr{O}(3\sigma(T)) \otimes \pi^*\mathscr{N}$ for some invertible sheaf $\mathscr{N}$. An isomorphism of diagrams is an isomorphism of the families of curves which respects the sections and the embeddings.

(a) Show that $\mathscr{S}$ is a stack in the big étale topology on $\operatorname{Spec}\mathbb{Z}$ with discrete fiber groupoids, and hence is equivalent to a sheaf S.

(b) (This part uses cohomology and base change. Feel free to just read it if you are not familiar with the techniques used below.) Let $g : X \to B$ be a proper flat morphism of finite presentation (or you can let B be Noetherian if you want and forget about the finite presentation hypothesis) and $\mathscr{L}$ an invertible sheaf on X. Consider the complex $\mathbf{R}g_*\mathscr{L}$. We can represent it locally on B as a complex of free modules of the form $F^0 \to F^1 \to \ldots$ in such a way that the base change of this complex along $B' \to B$ computes $\mathbf{R}(g_{B'})_*\mathscr{L}_{B'}$. Viewing $F^0 \to F^1$ as a matrix with functions in B, use determinants to define a subscheme $B_0 \subset B$ with the property that a map $Q \to B$ factors through B_0 if and only if every geometric fiber of $\mathscr{L}$ over Q has a non-zero section. Doing the same thing for $\mathscr{L}^\vee$, show that there is a closed subscheme $Z \subset B$ such that a map $Q \to B$ factors through Z if and only if there exists an invertible sheaf $\mathscr{Q}$ on Q such that $\mathscr{L}|_{X\times_B Q} \cong (g_Q)^*\mathscr{Q}$.

(b) Use the preceding part to show that S is representable by a scheme F. Observe that S has a natural action of PGL_3 induced by the change of coordinates on $\mathbb{P}^2$.

(c) Given a scheme T and a family $(E, \sigma) \in (\mathscr{M}_{1,1})_T$, show that there is a natural diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & \mathbb{P}(\pi_*\mathscr{O}(3\sigma(T))) \\ {\scriptstyle\sigma}\uparrow\downarrow{\scriptstyle\pi} & \swarrow & \\ T & & \end{array}$$

arising from the natural surjection $\pi^*\pi_*\mathscr{O}(3\sigma(T)) \to \mathscr{O}(3\sigma(T))$. Let $I \to T$ be the PGL_3-torsor $\mathrm{Isom}(P^2, \mathbb{P}(\pi_*\mathscr{O}(3\sigma(T))))$. Show that there is an equivariant map $I \to S$ by observing that the pullback fo $\mathbb{P}(\pi_*\mathscr{O}(3\sigma(T)))$ to I is non-canonically isomorphic to $\mathbb{P}^2$ (thus giving rise to a map to S, whose equivariance you must now check). This defines a map of stacks $\epsilon : \mathscr{M}_{1,1} \to [S/\operatorname{PGL}_3]$.

(d) Show that for any x and y in $(\mathscr{M}_{1,1})_T$, the induced map $\operatorname{Isom}(x, y) \to \operatorname{Isom}(\epsilon(x), \epsilon(y))$ is a bijection, so that the map on fiber categories is fully faithful, and that for any scheme T and any object $z \in [S/\operatorname{PGL}_3]_T$, there is a covering $T' \to T$ and an object $y \in \mathscr{M}_{1,1}$ such that $x|_{T'} \cong \epsilon(x)$.

(e) Prove a general fact: suppose $\rho : \mathscr{X} \to \mathscr{Y}$ is a map of stacks on a site $\mathcal{C}$ such that (1) for any T, the functor $\rho_T : \mathscr{X}_T \to \mathscr{Y}_T$ is fully faithful, and (2) for any T and any $y \in \mathscr{Y}_T$, there is a covering $T' \to T$ and $x \in \mathscr{X}_{T'}$ such that $y|_{T'}$ is isomorphic to $\rho(x)$. Show that ρ is an equivalence.

(f) Thus, ϵ from part (d) is an equivalence, which shows that $\mathscr{M}_{1,1}$ is a quotient stack (and then, since PGL_3 is smooth, that it has a representable smooth cover by a scheme). Try to produce an inverse to ϵ directly. (This is not meant to be a menacing suggestion.)

Exercise 7.3. Consider a cartesian diagram of schemes

$$\begin{array}{ccc} X_0 & \overset{i}{\hookrightarrow} & X \\ \downarrow & & \downarrow \\ S_0 & \overset{j}{\hookrightarrow} & S \end{array}$$

where j is a square-zero closed immersion defined by an ideal I, and X is flat over S. Let F_0 be a quasi-coherent sheaf on X_0 flat over S_0.

(a) Show that the "change-of-rings" spectral sequence

$$E_2^{pq} = \operatorname{Ext}^q_{\mathscr{O}_{X_0}}(\operatorname{Tor}_p^{\mathscr{O}_X}(F_0, \mathscr{O}_{X_0}), F_0 \otimes I) \implies \operatorname{Ext}_X^{p+q}(F_0, F_0 \otimes I)$$

induces a long exact sequence

$$0 \to \operatorname{Ext}^1_{\mathscr{O}_{X_0}}(F_0, F_0 \otimes I) \to \operatorname{Ext}^1_{\mathscr{O}_X}(F_0, F_0 \otimes I) \to \operatorname{Hom}(F_0 \otimes I, F_0 \otimes I) \xrightarrow{\delta} \operatorname{Ext}^2_{\mathscr{O}_{X_0}}(F_0, F_0 \otimes I).$$

(You may find the isomorphism $\operatorname{Tor}_1^{\mathscr{O}_X}(F_0, \mathscr{O}_{X_0}) \cong F_0 \otimes I$ useful.)

(b) Let $o \in \operatorname{Ext}^2_{\mathscr{O}_{X_0}}(F_0, F_0 \otimes I)$ denote the image of the identity map under δ. Show that $o = 0$ if and only if there exists a quasi-coherent sheaf F on X flat over S and an isomorphism $i^*F \to F_0$.

(c) Let S denote the set of isomorphism classes of pairs (F, σ), where F is a quasi-coherent sheaf on X flat over S and $\sigma : i^*F \to F_0$ is an isomorphism. Show that if $o = 0$ then the set S is a torsor under $\operatorname{Ext}^1_{\mathscr{O}_{X_0}}(F_0, F_0 \otimes I)$.

(d) Show that for any lifting (F, ι) the group of automorphisms is canonically isomorphic to $\operatorname{Ext}^0_{\mathscr{O}_{X_0}}(F_0, F_0 \otimes I)$.

Exercise 7.4. Let X be a scheme and let $\mathscr{P}ic$ be the Picard stack of line bundles on X. Show that $\mathscr{P}ic$ is isomorphic to $\operatorname{ch}(\mathscr{O}_X^* \to 0)$.

Exercise 7.5. Let $S_0 \hookrightarrow S$ be a closed immersion defined by a square-zero ideal J and let $X_0 \to S_0$ be a smooth morphism. Let Def_{X_0} denote the stack over $|X_0|$ which to any open $U_0 \subset X_0$ associates the groupoid of smooth liftings of U_0 to S. Show that Def_{X_0} is a Picard stack and equivalent to $\operatorname{ch}(T_{X_0/S_0} \otimes J \to 0)$.