

EXERCISES 8

Exercise 8.1. Let X_Λ be a scheme over Λ , and \mathcal{E}_Λ a coherent sheaf on X_Λ . Fix X and \mathcal{E} to be the restrictions to k , and let \mathcal{F} be a quotient of \mathcal{E} , with kernel \mathcal{G} . Suppose further that \mathcal{E}_Λ is flat over Λ . Then $\text{Def}_{\mathcal{F}, \mathcal{E}}$ has an obstruction theory taking values in $\text{Ext}^1(\mathcal{G}, \mathcal{F})$.

More generally, if one takes the global functor on Sch_S of quotients of a coherent sheaf \mathcal{E}_X on some X/S , with \mathcal{E}_X flat over S , then there is an obstruction theory in the sense of Artin coming from $\text{Ext}^1(\mathcal{G}, \mathcal{F})$.

Exercise 8.2. Let X_Λ be a scheme over Λ , and X the restriction to k . Fix a closed subscheme $Z \subseteq X$. Then the tangent space to $\text{Def}_{Z, X}$ is $\underline{\text{Hom}}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z)$, with an obstruction theory taking values in $\text{Ext}_X^1(\mathcal{I}_Z, \mathcal{O}_Z)$.

Finally, if Z is a local complete intersection inside X , and X is smooth, show directly that all deformations are locally unobstructed, the tangent space is given by $H^0(Z, \mathcal{N}_{Z/X})$ and there is an obstruction theory taking values in $H^1(Z, \mathcal{N}_{Z/X})$.

Exercise 8.3. Let X_Λ and Y_Λ be schemes smooth over Λ , and Y_Λ separated, and let X and Y be the restrictions to k . Given a morphism $f : X \rightarrow Y$, show that the tangent space to Def_f is given by $H^0(X, f^*T_Y)$, and there is an obstruction theory

Exercise 8.4. (A solution to all of our problems)

Here is the statement of (our version of) Artin's theorem. Let S be a scheme locally of finite type over an excellent Dedekind scheme. (Note: using results proven since Artin's paper was written, one can actually simply assume that S is excellent and dispense with the Dedekind scheme. See the unique collaborative paper by Brian Conrad and Johan de Jong for more information.)

Let \mathcal{X} be a stack locally of finite presentation over S_{ET} with an obstruction theory \mathcal{O} (in the sense of Martin's lecture). Given an object $a \in \mathcal{X}_T$ and a map $g : T \rightarrow T'$, we get a groupoid $\mathcal{X}_a(T') = \{b \in \mathcal{X}(T'), \phi : a \xrightarrow{\sim} g^*b\}$. Given a ring A with a map $\text{Spec } A \rightarrow S$, we will also write $\mathcal{X}(A)$ for $\mathcal{X}_{\text{Spec } A}$. Thus, a ring map $A \rightarrow B$ (over \mathcal{O}_S) yields a (non-canonical) base change functor $\mathcal{X}(A) \rightarrow \mathcal{X}(B)$.

The following generalized Schlessinger conditions came up in the lecture.

(S1') Given a deformation situation $A' \rightarrow A \rightarrow A_0$ with A_0 reduced and of finite type over S , and given $B \rightarrow A$ such that $B \rightarrow A \rightarrow A_0$ is surjective, the functor

$$\mathcal{X}_a(A' \times_A B) \rightarrow \mathcal{X}_a(A') \times \mathcal{X}_a(B)$$

is an equivalence of categories for any $a \in \mathcal{X}(A)$.

(S2) Given a finite A_0 -module, the A_0 -module $D_{a_0}(A_0[M])$ is a finite A_0 -module for all $a_0 \in \mathcal{X}(A_0)$.

We also had some nice conditions on the deformation and obstruction theory. Fix a deformation situation $A' \rightarrow A \rightarrow A_0$ and $a \in \mathcal{X}(A_0)$. Lower case letters will denote the base change of a to various rings (which will be obvious from context).

(4.1) (i) If $p : A \rightarrow B$ is étale, then the natural maps

$$D_{a_0}(M) \otimes B_0 \rightarrow D_{b_0}(M \otimes B_0)$$

and

$$\mathcal{O}_a(M) \otimes B_0 \rightarrow \mathcal{O}_b(M \otimes B_0)$$

are isomorphisms.

(ii) If $\mathfrak{c} \subset A_0$ is a maximal ideal then the natural map

$$D_{a_0}(M) \otimes \widehat{A}_0 \rightarrow \varprojlim D_{a_0}(M/\mathfrak{m}^n M)$$

is an isomorphism.

(iii) There is a dense set of closed points $p \in \text{Spec } A_0$ such that the natural map

$$D_{a_0}(M) \otimes \kappa(p) \rightarrow D_{a_0}(M \otimes \kappa(p))$$

is an isomorphism and the natural map

$$\mathcal{O}_a(M) \otimes \kappa(p) \rightarrow \mathcal{O}_a(M \otimes \kappa(p))$$

is injective.

Theorem (Artin). *Let \mathcal{X} be a stack locally of finite presentation on S_{ET} . Then \mathcal{X} is an Artin stack locally of finite type over S if*

(1) *(S1') and (S2) hold and if $a_0 \in \mathcal{X}(A_0)$ (A_0 of finite type over S), then $\text{Aut}_{a_0}^{\text{inf}}(A_0[M])$ is a finite A_0 -module.*

(2) *For any complete local \mathcal{O}_S -algebra \widehat{A} with residue field finite over S , the natural functor*

$$\mathcal{X}(\widehat{A}) \rightarrow \varprojlim \mathcal{X}(\widehat{A}/\mathfrak{m}^n)$$

is an equivalence.

(3) *D and \mathcal{O} satisfy (4.1) and $\text{Aut}_{a_0}^{\text{inf}}(A_0[M])$ satisfies the same conditions (4.1) as does D .*

(4) *If A_0 is of finite type over S and $a_0 \in \mathcal{X}(A_0)$, then any $\phi \in \text{Aut}(a_0)$ which agrees with the identity map over a dense set of points of $\text{Spec } A_0$ must equal the identity.*

(5) *The diagonal of \mathcal{X} (whose representability follows from the first four parts) is quasi-compact.*

Apply Artin's theorem (except where indicated) to show that the following are Artin stacks locally of finite type over S . (Note: compare this list to the problems we started talking about on the first day of the workshop.)

(a) The stack \mathfrak{V} of canonically polarized varieties: the fiber category \mathfrak{V}_T consist of proper smooth families $V \rightarrow T$ such that $\omega_{V/T}$ is T -ample. (Recall that the deformation theory is governed by the cohomology of the relative tangent sheaf.)

(b) The stack \mathcal{M}_g of smooth curves of genus g with $g \neq 1$. (If you allow the families to be algebraic spaces, then $g = 1$ is allowed.)

(c) Let $f : X \rightarrow S$ be a proper flat morphism. Show that the stack $\mathcal{P}ic_{X/S}$ whose fiber category over T consists of invertible sheaves \mathcal{L} on $X \times_S T$ is an Artin stack locally of finite type over S . (Is it of finite type? Is it separated?) (You may be used to seeing that additional

hypothesis that for all $T \rightarrow S$, the natural map $\mathcal{O}_T \rightarrow (f_T)_*(\mathcal{O}_{X_T})$ is an isomorphism [this is called being cohomologically flat in degree 0] when “algebraic” is used in conjunction with “Picard.” This hypothesis is only necessary to ensure that the sheafification of the Picard stack – classically called the Picard functor – is an algebraic space. If you happen to know what a gerbe is, the cohomological flatness hypothesis ensures that $\mathcal{P}ic_{X/S}$ is a \mathbf{G}_m -gerbe over its sheafification, from which it follows that the sheafification is an algebraic space. There are other ways to phrase this.)

(d) Let X/S be of finite type and fix a sheaf \mathcal{F} on X . Let $\text{Quot}_{X/S}(\mathcal{F})$ be the stack whose fiber category over $T \rightarrow S$ consists of quotient maps $\mathcal{F} \rightarrow \mathcal{Q}$ with \mathcal{Q} a T -flat quasi-coherent sheaf of finite presentation. Show that $\text{Quot}_{X/S}(\mathcal{F})$ is an Artin stack locally of finite type over S . (As a special case, consider what happens with X is proper and flat and $\mathcal{F} = \mathcal{O}$. This is good enough for many applications.)

(e) Let X and Y be two S schemes of finite type with X proper and flat and Y separated. Show that the stack (functor) of maps from X to Y is an Artin stack ([quasi-separated] algebraic space) locally of finite type over S . Avoid Artin’s theorem: the graph of a map $X \rightarrow Y$ is identified with a closed subscheme, proper and flat over the base, which projects isomorphically to X . On the other hand, a closed subscheme is identified with a quotient of the structure sheaf. Thus, show that this problem is open in (special case of) the problem from part (d).

(f) Show that subspaces of a fixed vector space form a separated algebraic space using part (d).

Exercise 8.5. Consider a commutative diagram of schemes

$$\begin{array}{ccc} X & \xrightarrow{j} & P \\ & \searrow f & \downarrow g \\ & & S, \end{array}$$

where g is smooth and j is an immersion. Let I denote the ideal of the closure of X in P so that we have the complex

$$L^{(P)} : j^*I \rightarrow j^*\Omega_{P/S}^1.$$

The aim of this exercise is to show that if

$$\begin{array}{ccc} X & \xrightarrow{j'} & P' \\ & \searrow f & \downarrow g' \\ & & S, \end{array}$$

is a second factorization with g' smooth then there is a canonical quasi-isomorphism between $L^{(P)}$ and the complex $L^{(P')}$ obtained from P' .

(a) By considering the product $P \times_S P'$, reduce to showing that if we have a commutative diagram

$$\begin{array}{ccc}
 & & P' \\
 & \nearrow^{j'} & \downarrow h \\
 X & \xrightarrow{j} & P \\
 & \searrow f & \downarrow g \\
 & & S,
 \end{array}
 \quad \begin{array}{c} \\ \\ \\ \\ \curvearrowright g' \end{array}$$

then the natural map of complexes $\varphi : L^{(P)} \rightarrow L^{(P')}$ is a quasi-isomorphism (in fact here it even suffices to consider just the case when h is smooth).

(b) For a two-term complex $F \rightarrow J$ of \mathcal{O}_X -modules, define $\underline{\text{Exal}}_S(X, F \rightarrow J)$ to be the kernel of the natural morphism of Picard stacks

$$\underline{\text{Exal}}_S(X, F) \rightarrow \underline{\text{Exal}}_S(X, J).$$

Show that

$$\underline{\text{Exal}}_S(X, F \rightarrow J) \simeq \text{ch}(\tau_{\leq 1} \mathcal{R}\mathcal{H}om(L^{(P)}, F \rightarrow J)[1]).$$

(Hint: Use the earlier exercise describing the kernel in terms of a cone).

(c) Let h_P denote the composite functor

$$D^{[-1,0]}(\mathcal{O}_X) \xrightarrow{\tau_{\leq 1} \mathcal{R}\mathcal{H}om(L^{(P)}, -)[1]} D^{[-1,0]}(\mathcal{O}_X) \xrightarrow{H^1} (\text{Groups})$$

and define $h_{P'}$ similarly. Show that φ induces an isomorphism of functors $h_{P'} \rightarrow h_P$ and conclude by Yoneda's lemma that φ is a quasi-isomorphism.

(d) Prove directly (i.e. without using Picard stacks) that φ is a quasi-isomorphism.