EXERCISES 8

Exercise 8.1. Let X_Λ be a scheme over Λ, and $\mathcal{E}_\Lambda$ a coherent sheaf on X_Λ. Fix X and $\mathcal{E}$ to be the restrictions to k, and let $\mathcal{F}$ be a quotient of $\mathcal{E}$, with kernel $\mathcal{G}$. Suppose further that $\mathcal{E}_\Lambda$ is flat over Λ. Then $\mathrm{Def}_{\mathcal{F},\mathcal{E}}$ has an obstruction theory taking values in $\mathrm{Ext}^1(\mathcal{G},\mathcal{F})$.

More generally, if one takes the global functor on Sch_S of quotients of a coherent sheaf $\mathcal{E}_X$ on some X/S, with $\mathcal{E}_X$ flat over S, then there is an obstruction theory in the sense of Artin coming from $\mathrm{Ext}^1(\mathcal{G},\mathcal{F})$.

Exercise 8.2. Let X_Λ be a scheme over Λ, and X the restriction to k. Fix a closed subscheme $Z \subseteq X$. Then the tangent space to $\mathrm{Def}_{Z,X}$ is $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{I}_Z,\mathcal{O}_Z)$, with an obstruction theory taking values in $\mathrm{Ext}^1_X(\mathcal{I}_Z,\mathcal{O}_Z)$.

Finally, if Z is a local complete intersection inside X, and X is smooth, show directly that all deformations are locally unobstructed, the tangent space is given by $H^0(Z,\mathcal{N}_{Z/X})$ and there is an obstruction theory taking values in $H^1(Z,\mathcal{N}_{Z/X})$.

Exercise 8.3. Let X_Λ and Y_Λ be schemes smooth over Λ, and Y_Λ separated, and let X and Y be the restrictions to k. Given a morphism $f : X \to Y$, show that the tangent space to Def_f is given by $H^0(X, f^*T_Y)$, and there is an obstruction theory

Exercise 8.4. (A solution to all of our problems)

Here is the statement of (our version of) Artin's theorem. Let S be a scheme locally of finite type over an excellent Dedekind scheme. (Note: using results proven since Artin's paper was written, one can actually simply assume that S is excellent and dispense with the Dedekind scheme. See the unique collaborative paper by Brian Conrad and Johan de Jong for more information.)

Let $\mathscr{X}$ be a stack locally of finite presentation over S_{ET} with an obstruction theory $\mathcal{O}$ (in the sense of Martin's lecture). Given an object $a \in \mathscr{X}_T$ and a map $g : T \to T'$, we get a groupoid $\mathscr{X}_a(T') = \{b \in \mathscr{X}(T'), \phi : a \xrightarrow{\sim} g^*b\}$. Given a ring A with a map $\mathrm{Spec}\, A \to S$, we will also write $\mathscr{X}(A)$ for $\mathscr{X}_{\mathrm{Spec}\, A}$. Thus, a ring map $A \to B$ (over $\mathscr{O}_S$) yields a (non-canonical) base change functor $\mathscr{X}(A) \to \mathscr{X}(B)$.

The following generalized Schlessinger conditions came up in the lecture.

(S1′) Given a deformation situation $A' \to A \to A_0$ with A_0 reduced and of finite type over S, and given $B \to A$ such that $B \to A \to A_0$ is surjective, the functor

$$\mathscr{X}_a(A' \times_A B) \to \mathscr{X}_a(A') \times \mathscr{X}_a(B)$$

is an equivalence of categories for any $a \in \mathscr{X}(A)$.

(S2) Given a finite A_0-module, the A_0-module $D_{a_0}(A_0[M])$ is a finite A_0-module for all $a_0 \in \mathscr{X}(A_0)$.

We also had some nice conditions on the deformation and obstruction theory. Fix a deformation situation $A' \to A \to A_0$ and $a \in \mathscr{X}(A_0)$. Lower case letters will denote the base change of a to various rings (which will be obvious from context).

(4.1) (i) If $p : A \to B$ is étale, then the natural maps

$$D_{a_0}(M) \otimes B_0 \to D_{b_0}(M \otimes B_0)$$

and

$$\mathcal{O}_a(M) \otimes B_0 \to \mathcal{O}_b(M \otimes B_0)$$

are isomorphisms.

(ii) If $\subset A_0$ is a maximal ideal then the natural map

$$D_{a_0}(M) \otimes \widehat{A}_0 \to \varprojlim D_{a_0}(M/\mathfrak{m}^n M)$$

is an isomorphism.

(iii) There is a dense set of closed points $p \in \operatorname{Spec} A_0$ such that the natural map

$$D_{a_0}(M) \otimes \kappa(p) \to D_{a_0}(M \otimes \kappa(p))$$

is an isomorphism and the natural map

$$\mathcal{O}_a(M) \otimes \kappa(p) \to \mathcal{O}_a(M \otimes \kappa(p))$$

is injective.

Theorem (Artin)**.** *Let $\mathscr{X}$ be a stack locally of finite presentation on S_{ET}. Then $\mathscr{X}$ is an Artin stack locally of finite type over S if*

(1) (S1') and (S2) hold and if $a_0 \in \mathscr{X}(A_0)$ (A_0 of finite type over S), then $\operatorname{Aut}^{inf}_{a_0}(A_0[M])$ is a finite A_0-module.

(2) For any complete local $\mathscr{O}_S$-algebra $\widehat{A}$ with residue field finite over S, the natural functor

$$\mathscr{X}(\widehat{A}) \to \varprojlim \mathscr{X}(\widehat{A}/\mathfrak{m}^n)$$

is an equivalence.

(3) D and $\mathcal{O}$ satisfy (4.1) and $\operatorname{Aut}^{inf}_{a_0}(A_0[M])$ satisfies the same conditions (4.1) as does D.

(4) If A_0 is of finite type over S and $a_0 \in \mathscr{X}(A_0)$, then any $\phi \in \operatorname{Aut}(a_0)$ which agrees with the identity map over a dense set of points of $\operatorname{Spec} A_0$ must equal the identity.

(5) The diagonal of $\mathscr{X}$ (whose representability follows from the first four parts) is quasi-compact.

Apply Artin's theorem (except where indicated) to show that the following are Artin stacks locally of finite type over S. (Note: compare this list to the problems we started talking about on the first day of the workshop.)

(a) The stack $\mathfrak{V}$ of canonically polarized varieties: the fiber category $\mathfrak{V}_T$ consist of proper smooth families $V \to T$ such that $\omega_{V/T}$ is T-ample. (Recall that the deformation theory is governed by the cohomology of the relative tangent sheaf.)

(b) The stack $\mathscr{M}_g$ of smooth curves of genus g with $g \neq 1$. (If you allow the families to be algebraic spaces, then $g = 1$ is allowed.)

(c) Let $f : X \to S$ be a proper flat morphism. Show that the stack $\mathscr{P}ic_{X/S}$ whose fiber category over T consists of invertible sheaves $\mathscr{L}$ on $X \times_S T$ is an Artin stack locally of finite type over S. (Is it of finite type? Is it separated?) (You may be used to seeing that additional hypothesis that for all $T \to S$, the natural map $\mathscr{O}_T \to (f_T)_*(\mathscr{O}_{X_T})$ is an isomorphism [this is called being cohomologically flat in degree 0] when "algebraic" is used in conjunction with "Picard." This hypothesis is only necessary to ensure that the sheafification of the Picard stack – classically called the Picard functor – is an algebraic space. If you happen to know what a gerbe is, the cohomological flatness hypothesis ensures that $\mathscr{P}ic_{X/S}$ is a $\mathbf{G}_m$-gerbe over its sheafification, from which it follows that the sheafification is an algebraic space. There are other ways to phrase this.)

(d) Let X/S be of finite type and fix a sheaf $\mathscr{F}$ on X. Let $\mathrm{Quot}_{X/S}(\mathscr{F})$ be the stack whose fiber category over $T \to S$ consists of quotient maps $\mathscr{F} \to \mathscr{Q}$ with $\mathscr{Q}$ a T-flat quasi-coherent sheaf of finite presentation. Show that $\mathrm{Quot}_{X/S}(\mathscr{F})$ is an Artin stack locally of finite type over S. (As a special case, consider what happens with X is proper and flat and $\mathscr{F} = \mathscr{O}$. This is good enough for many applications.)

(e) Let X and Y be two S schemes of finite type with X proper and flat and Y separated. Show that the stack (functor) of maps from X to Y is an Artin stack ([quasi-separated] algebraic space) locally of finite type over S. Avoid Artin's theorem: the graph of a map $X \to Y$ is identified with a closed subscheme, proper and flat over the base, which projects isomorphically to X. On the other hand, a closed subscheme is identified with a quotient of the structure sheaf. Thus, show that this problem is open in (special case of) the problem from part (d).

(f) Show that subspaces of a fixed vector space form a separated algebraic space using part (d).

Exercise 8.5. Consider a commutative diagram of schemes

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & P \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g} \\ & & S, \end{array}$$

where g is smooth and j is an immersion. Let I denote the ideal of the closure of X in P so that we have the complex

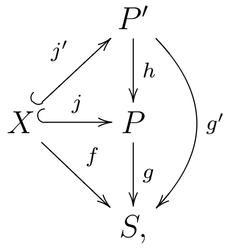
$$L^{(P)} : j^*I \to j^*\Omega^1_{P/S}.$$

The aim of this exercise is to show that if

$$\begin{array}{ccc} X & \overset{j'}{\hookrightarrow} & P' \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g'} \\ & & S, \end{array}$$

is a second factorization with g' smooth then there is a canonical quasi-isomorphism between $L^{(P)}$ and the complex $L^{(P')}$ obtained from P'.

(a) By considering the product $P \times_S P'$, reduce to showing that if we have a commutative diagram



then the natural map of complexes $\varphi : L^{(P)} \to L^{(P')}$ is a quasi-isomorphism (in fact here it even suffices to consider just the case when h is smooth).

(b) For a two-term complex $F \to J$ of $\mathscr{O}_X$-modules, define $\underline{\mathrm{Exal}}_S(X, F \to J)$ to be the kernel of the natural morphism of Picard stacks

$$\underline{\mathrm{Exal}}_S(X, F) \to \underline{\mathrm{Exal}}_S(X, J).$$

Show that

$$\underline{\mathrm{Exal}}_S(X, F \to J) \simeq \mathrm{ch}(\tau_{\leq 1}\mathscr{R}\mathscr{H}om(L^{(P)}, F \to J)[1]).$$

(Hint: Use the earlier exercise describing the kernel in terms of a cone).

(c) Let h_P denote the composite functor

$$D^{[-1,0]}(\mathscr{O}_X) \xrightarrow{\tau_{\leq 1}\mathscr{R}\mathscr{H}om(L^{(P)},-)[1]} D^{[-1,0]}(\mathscr{O}_X) \xrightarrow{H^1} (\text{Groups})$$

and define $h_{P'}$ similarly. Show that φ induces an isomorphism of functors $h_{P'} \to h_P$ and conclude by Yoneda's lemma that φ is a quasi-isomorphism.

(d) Prove directly (i.e. without using Picard stacks) that φ is a quasi-isomorphism.