Deformation Theory and Moduli in Algebraic Geometry

Deformations (b): representability and Schlessinger's criterion

Brian Osserman

1. Functors of Artin rings

We have already seen that a scheme X can be reconstructed from its functor of points; that is, from the data of all morphisms from all schemes to X. On a more restricted level, given a k-valued point $x \in X$ we have seen that with mild hypotheses the tangent space to X at x can be recovered from the set of morphisms $\operatorname{Spec} k[\epsilon] \to X$ with image x. The first gives a very global picture, while the second is as local as possible.

This lecture series will focus on something in between: functors of (local) Artin rings. From the point of view of moduli theory, we will be studying families over Artin rings. It may seem like not much is going on here, since geometrically everything is happening over a single point, but the infinitesimal "thickenings" we study turn out to carry a surprisingly rich trove of information, essentially equivalent to the "complete local rings" of the moduli space. The main focus of the lectures will be two notions of representability for functors of Artin rings, and a practical criterion developed by Schlessinger for testing representability.

1.1. Recovering complete local rings.

We begin by making precise the data one can recover from a scheme X by considering maps $\operatorname{Spec} A \to X$, where A is an Artin local ring. We use the following temporary notation:

Temporary Notation 1.1.1. Given a field k, let $\operatorname{Art}(k)$ denote the category of Artin local rings with residue field k, and with morphisms commuting with the surjection to k.

Given a locally Noetherian scheme X, and a point $x \in X$, let $F_{X,x} : \operatorname{Art}(k(x)) \to \operatorname{Set}$ be the covariant functor sending an Artin local ring A to the set of morphisms $\operatorname{Spec} A \to X$ such that the composition with $\operatorname{Spec} k(x) \to \operatorname{Spec} A$ gives the canonical map $\operatorname{Spec} k(x) \to X$ induced by x.

We have seen that when X is over a field k, and $k(x) = k$, then we have $F_{X,x}(k(x)[\epsilon])$ in bijection with the tangent space of X at x. The assertion is that the full data of the functor precisely recovers the complete local ring $\hat{\mathcal{O}}_{X,x}$.

Proposition 1.1.2. *The canonical map* $\operatorname{Spec} \hat{\mathcal{O}}_{X,x} \to X$ *induces an isomorphism*

$$F_{\operatorname{Spec} \hat{\mathcal{O}}_{X,x},x} \xrightarrow{\sim} F_{X,x},$$

and any complete local Noetherian ring R *with* $\operatorname{Spec} R \to X$ *inducing such a bijection is canonically isomorphic to* $\hat{\mathcal{O}}_{X,x}$.

Proof. The first statement is equivalent to the assertion that any map $f : \operatorname{Spec} A \to X$ in $F_{X,x}(A)$ factors uniquely through $\operatorname{Spec} \hat{\mathcal{O}}_{X,x}$.

We first observe that if A is any local ring, then any map $f : \operatorname{Spec} A \to X$ having the closed point of A mapping to x necessarily factors uniquely through the natural map $\operatorname{Spec} \mathcal{O}_{X,x} \to X$. Indeed, reduction to the affine case makes this an easy exercise, which we leave (informally) to the reader.

It then suffices to show that any local map of local rings $B \to A$ factors uniquely through $\hat{B}$ as long as B is Noetherian and A is Artin. Uniqueness is immediate from the injectivity of the map $B \to \hat{B}$, and we deduce existence from the following observation: the maximal ideal

$\mathfrak{m}_B$ of B maps into $\mathfrak{m}_A$, and since A is Artin, some power $\mathfrak{m}_A^n = 0$, so it follows that the map $B \to A$ factors through $B \to B/\mathfrak{m}_B^n$. We obtain the desired map $\hat{B} \to A$ by composing with $\hat{B} \to B/\mathfrak{m}_B^n$.

It remains to show the second statement, asserting in essence that $\hat{\mathcal{O}}_{X,x}$ is uniquely determined by the first statement. The point here is that $\hat{\mathcal{O}}_{X,x}/\mathfrak{m}_x^n$ and $R/\mathfrak{m}_R^n$ are both Artin local rings for all n, and by hypothesis we are given compatible bijections $\mathrm{Mor}(\mathcal{O}_{X,x}, A) \xrightarrow{\sim} \mathrm{Mor}(R, A)$ for every Artin local ring A. In particular, the canonical surjections $\hat{\mathcal{O}}_{X,x} \twoheadrightarrow \hat{\mathcal{O}}_{X,x}/\mathfrak{m}_x^n$ and $R \twoheadrightarrow R/\mathfrak{m}_R^n$ induce maps $\hat{\mathcal{O}}_{X,x} \twoheadrightarrow R/\mathfrak{m}_R^n$ and $R \twoheadrightarrow \hat{\mathcal{O}}_{X,x}/\mathfrak{m}_x^n$ for all n. One checks that the hypothesized compatibilities mean that these maps fit together to give homomorphisms $\hat{\mathcal{O}}_{X,x} \to R$ and $R \to \hat{\mathcal{O}}_{X,x}$, which are mutually inverse, giving the desired assertion. □

The last statement of the proposition anticipates the concept of prorepresentability, which we will begin to investigate in the next lecture.

One of the most basic, but frequently important, pieces of data captured by the complete local rings is the dimension of X at x. More sharply, one can think of the complete local ring as describing the "singularity type" of x in X, or as capturing data similar to a neighborhood in the complex analytic topology. Both these points of view are reinforced by the Cohen structure theorem, of which a special case is:

Theorem 1.1.3. *Let X be a locally Noetherian scheme over a field k, having dimension n at a smooth point x. Then $\hat{\mathcal{O}}_{X,x} \cong k(x)[[x_1, \dots, x_n]]$.*

This says that the complete local rings at a smooth point of a scheme is determined simply by the dimension (and residue field), which one can think of as analogous to the fact that complex manifolds are characterized by having analytic neighborhoods isomorphic to open subsets of $\mathbb{C}^n$.

As one would naively picture from looking at small neighborhoods of a point, passing to the complete local ring can also have effects like separating the "branches" of an irreducible curve at a node:

Example 1.1.4. Let C be the irreducible nodal curve $y^2 = x^3 - x^2$ over a field k, and $P = (0, 0)$. Then $\hat{\mathcal{O}}_{C,P} \cong k[[s,t]]/(st)$, the complete local ring at the origin of $\operatorname{Spec} k[s,t]/(st)$.

Of course, the complete local ring remembers more than topological information:

Example 1.1.5. Although the projection to the y-axis of the cuspidal curve $y^2 = x^3$ is a homeomorphism, the complete local rings at the origin are not isomorphic. Indeed, the complete local ring of the cusp is $k[[x,y]]/(y^2 - x^3)$, which has a 2-dimensional Zariski cotangent space $\mathfrak{m}/\mathfrak{m}^2$ generated by x and y. In contrast, the axis has complete local ring $k[[t]]$ with 1-dimensional Zariski cotangent space generated by t.

1.2. **The functors of interest.** With this motivation, we now discuss the functors we will consider. For later convenience, we fix a field k and a complete Noetherian ring Λ with residue field k, and work with the category $\mathrm{Art}(\Lambda, k)$ of local Artin Λ-algebras with residue field k. You should think of this as working in a relative setting over $\operatorname{Spec} \Lambda$. Schlessinger works with the following functors (although the terminology is not standard):

Definition 1.2.1. A **predeformation functor** is a covariant functor $F : \mathrm{Art}(\Lambda, k) \to \mathrm{Set}$ such that $F(k)$ is the one-element set. The **tangent space** T_F of F is defined to be $F(k[\epsilon])$.

The restriction on $F(k)$ reflects the idea that we want to work locally at a fixed k-valued point. For a general predeformation functor, we have the unfortunate terminological situation that the tangent space is only a set. However, we've seen in Corollary 1.12 of Martin's lectures that under mild hypotheses the tangent space is in fact a vector space over k. Also, it is important to keep in mind that the tangent space is a relative one over $\operatorname{Spec} \Lambda$.

From a moduli point of view, the idea is to study *deformations* over Artin rings of a fixed object defined over k. We'll see more precisely how these arise in several concrete examples below.

Remark 1.2.2. Some words on Λ:

Note that if we start with a scheme X over k, and a k-valued point x, and we set $\Lambda = k$, then any map $\operatorname{Spec} A \to X$ with image x, makes A into a k-algebra, so the functor $F_{X,x}$ we considered earlier is closely related to a predeformation functor, differing only in that the category of Artin rings considered has fewer morphisms, since we require our maps to be maps of k-algebras. One can check that in this setting, if we instead took the predeformation functor we would still recover $\hat{\mathcal{O}}_{X,x}$ in the same sense as in Proposition 1.1.2.

For many deformation problems, anyone who is happy to work only over fields can set Λ to k, so that we are considering Artin local k-algebras with residue field k. Otherwise, if k is perfect of characteristic p one typically takes Λ to be the Witt vectors of k, thus allowing one to study deformations in mixed characteristic. Indeed, this is universal in this case; every complete Noetherian local ring with residue field k is canonically an algebra over the Witt vectors (see Proposition 10 of II, §5 of [6]).

In a more involved setting, one might be working with families over a given base, and Λ is then frequently a complete ring of the base scheme, or something along those lines.

2. Some examples, and the statement of Schlessinger's criterion

We begin by giving some examples of predeformation functors. We then describe two forms of representability, and give the statement of Schlessinger's necessary and sufficient criterion for either one to hold. At first (and thereafter) the criterion may appear technical and somewhat lacking in intuition, but in practice it is relatively straightforward to check, given the appropriate tools.

2.1. **Examples.** We introduce some important examples of predeformation functors. If we have a "good" moduli functor $\tilde{F} : \mathrm{Sch} \to \mathrm{Set}$, and an element $\eta_0 \in \tilde{F}(\operatorname{Spec} k)$, we can obtain a well-behaved predeformation functor F simply by setting $F(A) = \{\eta \in \tilde{F}(\operatorname{Spec} A) : \eta|_{\operatorname{Spec} k} = \eta_0\}$; that is, we consider families over $\operatorname{Spec} A$ which restrict to the chosen object η_0 over $\operatorname{Spec} k$. The following is an example of this.

Example 2.1.1. Deformations of a closed subscheme. Let X_Λ be a scheme over $\operatorname{Spec} \Lambda$, and X its restriction to $\operatorname{Spec} k$. Let $Z \subseteq X$ be a closed subscheme. The predeformation functor $\mathrm{Def}_{Z,X} : \mathrm{Art}(\Lambda, k) \to \mathrm{Set}$ is defined by sending A to the set of closed subschemes $Y_A \subseteq X_\Lambda|_{\operatorname{Spec} A}$ which are flat over A, and restrict to Z over $\operatorname{Spec} k$.

However, this doesn't work well when the objects parametrizing the moduli functor have automorphisms – an early hint that for moduli of objects with automorphisms, functors are not the most natural objects to work with. In this case, we rigidify somewhat, as in the following example.

Example 2.1.2. Deformations of an abstract scheme. Let X be a scheme over $\operatorname{Spec} k$. The predeformation functor $\mathrm{Def}_X : \mathrm{Art}(\Lambda, k) \to \mathrm{Set}$ is defined by sending A to the set of isomorphism classes of pairs (X_A, φ), where X_A is a scheme flat over A, and $\varphi : X \to X_A$ is a morphism inducing an isomorphism $X \xrightarrow{\sim} X_A|_k$. Two pairs (X_A, φ) and (X'_A, φ') are considered to be isomorphic if there is an isomorphism $X_A \xrightarrow{\sim} X'_A$ commuting with φ and φ'.

Remark 2.1.3. For the first functors, we don't need to consider pairs including a map (although we are certainly considering Y_A as a closed subscheme, and not an abstract scheme), because there is a notion of equality for subschemes, and not simply isomorphism. However, if we passed naively from a global moduli functor of schemes to a predeformation functor in the second case, we would obtain a slightly different functor from the functor Def_X described above: we would not remember the φ as part of the data, so we would send A simply to isomorphism classes of schemes flat over A whose restriction to k is isomorphic to X. This is subtly different from what we have defined, in that it is possible to have two pairs $(X_A, \varphi), (X'_A, \varphi')$ which are abstractly isomorphic as schemes, but do not admit an isomorphism commuting with φ and φ', for instance, if X has an automorphism not extending to X_A and X'_A. Thus, we see that the functor we have defined is more "rigid", and it will in general be better-behaved. See Theorem 8.18 of [2] for further discussion in the case of Def_X.

An example of a somewhat different flavor is the following:

Example 2.1.4. Deformations of a sheaf. Let X_Λ be a scheme over $\operatorname{Spec}\Lambda$, and X its restriction to $\operatorname{Spec} k$. Let $\mathcal{E}$ be a quasicoherent sheaf on X. The predeformation functor $\mathrm{Def}_{\mathcal{E}} : \mathrm{Art}(\Lambda, k) \to \mathrm{Set}$ is defined by sending A to the set of isomorphism classes of pairs $(\mathcal{E}_A, \varphi)$ with $\mathcal{E}_A$ a quasicoherent sheaf on $X_\Lambda|_{\operatorname{Spec} A}$, flat over A, and $\varphi : \mathcal{E}_A \to \mathcal{E}$ a map of $\mathcal{O}_{X_\Lambda|_{\operatorname{Spec} A}}$-modules which induces an isomorphism $\mathcal{E}_A \otimes_A k \xrightarrow{\sim} \mathcal{E}$. As with deformations of schemes, two pairs are isomorphic if there is an isomorphism $\mathcal{E}_A \xrightarrow{\sim} \mathcal{E}'_A$ commuting with φ and φ'.

In all cases above, functoriality is defined in the obvious way by pullback.

Remark 2.1.5. We conclude with some vague comments on flatness, which is an opaque, frequently frustrating, but extremely powerful condition. Each of the above examples imposes a flatness condition over A. The conceptual reason for this is reasonably clear: if we want to consider a family $\mathcal{X}$ over a base S, we intuitively want to have two properties: first, $\mathcal{X}$ should surject onto S in some strong sense, so that the family is naturally "over S" and not some smaller subscheme; and second, the fibers of $\mathcal{X}$ over S should "vary continuously" so that we can naturally consider them as a family parametrized by S.

Flatness accomplishes both these things in a relatively clean manner. While it can be understood in more concrete terms over, for instance a discrete valuation ring (it is simply equivalent to the condition that $\mathcal{X}$ has no associated points over the closed point), in general it is much subtler, particularly in the case of a non-reduced base, where it cannot be expressed so geometrically. Since we work extensively with non-reduced bases in deformation theory, we simply work the concept of flatness into all our basic deformation problems without further comment. We do however mention that over a local Artin ring, flatness is very concrete: it is equivalent to freeness.

2.2. **Prorepresentability and hulls.** The strongest form of representability we will consider for predeformation functors is not representability in the strict sense, but what is called "prorepresentability." The reason for this is that as we saw in the case of a predeformation functor obtained from maps to a scheme, the "representing" object is typically not necessarily an Artin ring, but rather a limit of Artin rings – or more specifically, a complete Noetherian local ring. There are two equivalent definitions of prorepresentability.

Definition 2.2.1. Given a covariant functor $F : \mathrm{Art}(\Lambda, k) \to \mathrm{Set}$, let $\widehat{\mathrm{Art}}(\Lambda, k)$ be the category of complete Noetherian local Λ-algebras with residue field k, and define the associated functor $\hat{F} : \widehat{\mathrm{Art}}(\Lambda, k) \to \mathrm{Set}$ by $\hat{F}(R) := \varprojlim_n F(R/\mathfrak{m}^n)$. We say that F is **prorepresentable** if and only if $\hat{F}$ is representable.

Note that our definition follows Schlessinger; Grothendieck used a different and somewhat more general definition.

Exercise 2.2.2. Show that F is prorepresentable if and only if there exists a complete Noetherian local ring R and a collection $\eta_n \in F(R/\mathfrak{m}^n)$ of elements, compatible with restriction, such that for every $A \in \mathrm{Art}(\Lambda, k)$, and every $\eta \in F(A)$, there exists a unique map $f_\eta : R \to A$ such that if f_η factors through $R/\mathfrak{m}^n$, we have that η is the image of η_n.

Warning 2.2.3. When we apply the above situation to moduli problems, it is frequently the case that for a complete ring R, the set $F(R)$ will *not* be the same as the set of families over $\operatorname{Spec} R$. It will rather correspond to a compatible collection of families over $\operatorname{Spec} R/\mathfrak{m}_R^n$ for all n, but whether these actually come from a family over $\operatorname{Spec} R$ is a subtle topic, called effectivization. This will be addressed in the background lecture on the Grothendieck existence theorem, and later on when we examine the issue of which deformations can always be effectivized.

As is the case with global moduli functors and representability, it turns out that prorepresentability is often too much to hope for. The next best situation to hope for is that we have what Schlessinger calls a "hull." In order to define a hull, we need to first define the notion of (formal) smoothness for a morphism of functors.

Definition 2.2.4. Let $F, F' : \mathrm{Art}(\Lambda, k) \to \mathrm{Set}$ be covariant functors, with a morphism $\varphi : F \to F'$. We say φ is **smooth** if for every surjection $A \twoheadrightarrow B$ in $\mathrm{Art}(\Lambda, k)$, the map

$$F(A) \to F(B) \times_{F'(B)} F'(A)$$

is surjective.

Notice that this is precisely the formal criterion for smoothness for morphisms of finite type between locally Noetherian schemes, translated into the abstract setting of functors.

Notation 2.2.5. Let $F : \mathrm{Art}(\Lambda, k) \to \mathrm{Set}$ be a covariant functor, and $\hat{F}$ as above. Given R a complete Noetherian local ring, we denote by $h_R : \widehat{\mathrm{Art}}(\Lambda, k) \to \mathrm{Set}$ the functor of (co)points of R, and by $\bar{h}_R$ its restriction to $\mathrm{Art}(\Lambda, k)$.

Definition 2.2.6. Let F be a predeformation functor, and $\hat{F}$ as above. We say a pair (R, η), with R a complete Noetherian local ring, and $\eta \in \hat{F}(R)$, is a **hull** for F if the induced map $\bar{h}_R \to F$ is smooth, and induces a bijection $T_{\bar{h}_R} \xrightarrow{\sim} T_F$ on tangent spaces.

We follow the terminology used by Schlessinger. Hulls are also sometimes called minimal versal deformation spaces or miniversal deformation spaces.

Note that prorepresentability of F is equivalent to the existence of a pair (R, η) as above with $\bar{h}_R \to F$ an isomorphism. Hulls can be thought of as a loose analogue of coarse moduli spaces.

By Yoneda's lemma, a pair (R, η) prorepresenting F is unique up to unique isomorphism. It turns out that hulls are unique up to *non*-unique isomorphism:

Proposition 2.2.7. *Let (R, η) and (R', η') be two hulls for a predeformation functor F. Then there exists an isomorphism $u : R \to R'$ such that $F(u)(\eta) = \eta'$.*

The proof is left as an exercise.

2.3. **Schlessinger's criterion.**

We need one more bit of terminology in order to state Schlessinger's criterion in its strongest form:

Definition 2.3.1. We say a surjective map $f : A \twoheadrightarrow B$ in $\mathrm{Art}(\Lambda, k)$ is a **small thickening** if $\ker f \mathfrak{m}_A = 0$ and $\ker f$ is principal; or equivalently, if $\ker f \cong k$.

Note that this is called a "small extension" by Schlessinger and many others, but not uniformly (e.g., [3] uses "small extension" differently). It is an easy exercise to check that every surjection in $\mathrm{Art}(\Lambda, k)$ can be factored as a series of small thickenings.

The statement of Schlessinger's criterion frequently refers to the following map, which we obtain from the definition of a functor whenever we are given $A' \to A$, $A'' \to A$ in $\mathrm{Art}(\Lambda, k)$, and a covariant functor F:

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''). \tag{1}$$

Schlessinger's theorem is then the following criterion:

Theorem 2.3.2. *Let F be a predeformation functor. Then F has a hull if and only if the following conditions (H1), (H2), (H3) are satisfied, and F is prorepresentable if and only if in addition (H4) is satisfied.*

- (H1) *The map* (1) *is surjective $A'' \to A$ is a small thickening.*
- (H2) *The map* (1) *is bijective whenever $A = k$ and $A'' = k[\epsilon]$.*
- (H3) *The tangent space T_F is finite-dimensional over k.*
- (H4) *The map* (1) *is bijective whenever $A'' = A'$ and $A' \to A$ is a small thickening.*

Remarks 2.3.3. Note that if (H2) is satisfied, then by Corollary 1.12 in Martin's lectures the tangent space in fact has a k-vector space structure, so (H3) makes sense.

The conditions (H1)-(H4) are stated as weakly as possible. However, they imply substantially stronger conditions. For instance, if F is prorepresentable, it is an easy exercise to verify that (1) is always a bijection. Since every surjection can be factored into small thickening, (H1) implies the same surjectivity condition for arbitrary surjections $A'' \to A$. Similarly, if (H2) is satisfied, then by inducting on the dimension of a vector space V, we conclude that the same bijectivity condition is satisfied for $A' = k[V]$, with V any k-vector space.

3. An example of Schlessinger's criterion

We introduce a notion of deformation functor, which is in practice always satisfied for natural deformation problems. We then examine deformations of abstract schemes and show that the corresponding functors are deformation functors.

3.1. **Deformation functors and a first example.** Generally speaking, we find that (H1) and (H2) are always satisfied for every reasonably natural deformation problem we can come up with, while (H3) tends to require properness conditions. (H4) is frequently not satisfied, and can be best understood in terms of automorphisms of objects being parametrized. However, because (H1) and (H2) are so ubiqitous, and also because if (H2) is satisfied we have that T_F is in fact a vector space, we make the following definition:

Definition 3.1.1. A **deformation functor** is a predeformation functor satisfying (H1) and (H2).

Our first example of a deformation functor will be Def_X, the functor of deformations of a scheme over k.

To give the sharpest statement, we define:

Definition 3.1.2. Given $(X_A, \varphi) \in \mathrm{Def}_X(A)$ an **automorphism** of (X_A, φ) (or **infinitesimal automorphism** of X_A) is an automorphism of X_A over A, commuting with φ.

Theorem 3.1.3. *Let X be a scheme over k, and Def_X the predeformation functor parametrizing flat deformations of X. Then:*

- (i) Def_X *satisfies (H1) and (H2), so is a deformation functor.*
- (ii) Def_X *satisfies (H3) if X is proper over k.*
- (iii) Def_X *satisfies (H4) if and only if for every pair $(X_{A'}, \varphi)$ over A' and $A' \to A$ a small thickening, every automorphism of $(X_{A'}|_A, \varphi_A)$ extends to an automorphism of $(X_{A'}, \varphi)$.*

Note that properness is not required for (H3) to hold: for instance, if X is smooth and affine we saw that the tangent space of Def_X is 0-dimensional!

Schlessinger's criterion then implies the following:

Corollary 3.1.4. *In the situation of the theorem, if X is proper, then Def_X has a hull, and if further $H^0(X, \mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X)) = 0$, then Def_X is prorepresentable.*

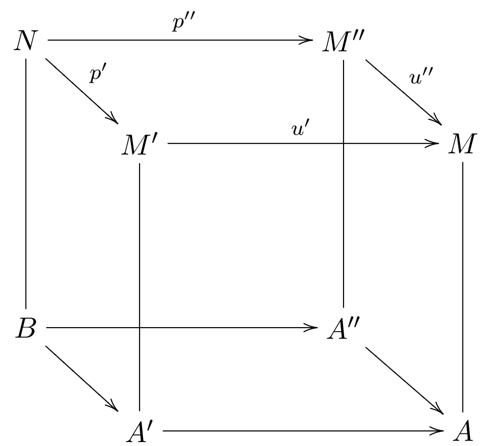
If X is smooth, then $\mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X)$ is simply T_X, so we see for instance the following:

Example 3.1.5. Suppose X is a smooth, proper curve of genus g. Then Def_X has a hull, and if $g \geqslant 2$ we have that Def_X is prorepresentable.

In fact, the restriction on genus is a bit of a red herring here, as it is still possible to have prorepresentability even in the presence of infinitesimal automorphisms, as long as the automorphisms always extend.

3.2. **Proof of the theorem.** To prove the theorem, we will use some general results on flat modules. We remark that if we restrict our attention only to local Artin rings, the proofs are simpler, since every flat module is free. However, for compatibility with future lectures, we prove a more general result.

Lemma 3.2.1. *(Schlessinger,* [5, Lemma 3.4]*) Consider a commutative diagram*



of compatible ring and module homomorphisms, where $B = A' \times_A A''$, $N = M' \times_M M''$, and M' and M'' are flat over A' and A'', respectively. Suppose

(i) *$A'' \twoheadrightarrow A$, with nilpotent kernel,*
(ii) *u' induces $M' \otimes_{A'} A \xrightarrow{\sim} M$, and similarly for u''.*

Then N is flat over B, and p' induces $N \otimes_B A' \xrightarrow{\sim} M'$, and similarly for p''.

Next, suppose that with the above notation, we also have a B-module L and maps $q' : L \to M'$ and $q'' : L \to M''$ such that q' induces $L \otimes_B A' \xrightarrow{\sim} M'$. Then the map $q' \times q'' : L \to N$ is an isomorphism.

This is left as an exercise.

We apply the lemmas to prove the following more general proposition:

Proposition 3.2.2. *Suppose we are given ring homomorphisms $A' \to A$ and $A'' \to A$, where $A'' \to A$ is surjective, with nilpotent kernel. Let $B = A' \times_A A''$. Then:*

(i) *Given schemes X' and X'' flat over A' and A'', and an isomorphism $\varphi : X'|_A \xrightarrow{\sim} X''|_A$, there exists a scheme Y flat over B and morphisms $\varphi' : X' \to Y$ and $\varphi'' : X'' \to Y$ inducing isomorphisms $X' \xrightarrow{\sim} Y|_{A'}$ and $X'' \xrightarrow{\sim} Y|_{A''}$, and such that $\varphi = \varphi''^{-1}|_A \circ \varphi'|_A$.*
(ii) *Given schemes Y_1, Y_2 flat over B, the natural map*

$$\mathrm{Isom}(Y_1, Y_2) \to \mathrm{Isom}(Y_1|_{A'}, Y_2|_{A'}) \times_{\mathrm{Isom}(Y_1|_A, Y_2|_A)} \mathrm{Isom}(Y_1|_{A''}, Y_2|_{A''})$$

is a bijection.

Proof. To check (i), we construct Y on the topological space of X'. If we identify the topological spaces of X'' and $X''|_A$, and use φ to identify both spaces with $X'|_A$, we denote by $i : X'|_A \to X'$ the map of underlying topological spaces, and we can set

$$\mathcal{O}_Y(U) = \mathcal{O}_{X'}(U) \times_{\mathcal{O}_{X'|_A}(i^{-1}(U))} \mathcal{O}_{X''}(i^{-1}(U)).$$

It follows immediately from the first part of Lemma 3.2.1 that, considered as modules over the base Artin rings, we have $\mathcal{O}_Y$ flat over B, and restricting to $\mathcal{O}_{X'}$ and $\mathcal{O}_{X''}$ over A' and A'' respectively, as desired. Finally, to check that $\mathcal{O}_Y$ defines a scheme structure Y, it suffices to check that fiber product commutes with localization, which is an easy exercise.

For (ii), we see immediately from the second part of Lemma 3.2.1 that the given map is surjective on the level of modules, and it is clear that a module isomorphism which is an algebra isomorphism after restriction to A' and A'' is necessarily an algebra isomorphism, since we now know that our modules over B are all isomorphic to fiber products of modules over A' and A''. This gives surjectivity of the map in question. But by the same token, since all our modules over B are isomorphic to fiber products of the restrictions to A' and A'', it is clear that homomorphisms in general and isomorphisms in particular are determined by their restrictions to A' and A''. □

Proof of theorem. It is easy to check that (i) follows from the above proposition. Indeed, (H1) is nearly a special case of (i) from the proposition, except that Def_X is defined with the additional data of the rigidifying map $\varphi : X \to X_A$. However, the fact that our morphisms in Def_X are required to commute with φ means that all the φ's simply come along for the ride: we get an induced map $\varphi_Y : X \to Y$ induced by $\varphi' : X \to X'$ and $\varphi'' : X \to X''$, which then restricts to φ' and φ'', as required.

We next conclude (H2) from (ii) of the proposition, more generally obtaining injectivity of (1) whenever $A = k$. In this case, the various maps φ ensure that any isomorphisms we have will restrict to the same isomorphism over A, so if we have two pairs over B, which are isomorphic over A' and A'', those isomorphisms necessarily agree on A, so by the surjectivity in (ii) come from an isomorphism of the original pairs over B, giving the necessary injectivity on isomorphism classes for (H2).

We next prove (iii), again using the proposition. (1) can fail to be injective only if we have (Y_1, φ_1) and (Y_2, φ_2) over B which are isomorphic after restriction to A' and A'', but not isomorphic. Suppose we have isomorphisms ψ' and ψ'' on $A' = A''$; by (ii) of the proposition, the only problem comes if ψ' and ψ'' cannot be chosen to give the same isomorphism after restriction to A. But the restrictions must differ by an automorphism of $(Y_1|_A, \varphi_1|_A)$, and if every such automorphism extends to an automorphism of (Y_1, φ_1), we can modify ψ' so that the restrictions to A agree. This shows that if every automorphism extends, (H4) is satisfied. But conversely, if there is an automorphism ψ over A which doesn't extend over A', we can start with $X' = X''$ over $A' = A''$, and choose our isomorphism over A to be either ψ or the identity map, and we will obtain non-isomorphic schemes over B which are isomorphic over A' and A''. This proves the first assertion of (iii).

Next, we claim that if $H^0(X, \mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X)) = 0$, then in fact all infinitesimal automorphisms are trivial. It then follows (see the remarks following Proposition 2.2 of Martin's lectures) that all infinitesimal automorphisms over $k[J]$ are trivial, and more generally that when $A' \twoheadrightarrow A$ with square-zero kernel, that there is a unique automorphism of any object over A' restricting to a given automorphism over A. If we take any $A \in \mathrm{Art}(\Lambda, k)$, writing the map $A \to k$ as a sequence of surjections with square-zero kernel, we obtain the assertion that every pair over A has only trivial automorphisms.

Finally, we note that (ii) is an immediate consequence of Proposition 2.6 of Martin's lectures when X is proper and smooth over k, since the tangent space is given by $H^1(X, T_X)$, and T_X

is locally free, hence coherent. More generally, one can show that the tangent space is always finite-dimensional when X is proper over k. See §8 of Martin's lectures, and in particular the coherence result in (i), as well as Theorem 8.4 (ii), for an approach using Illusie's cotangent complex. See also Remarks 5.6 and 3.11 of [2] for an argument using a truncated form of the cotangent complex due to Lichtenbaum and Schlessinger. □

4. The proof of Schlessinger's criterion

Our aim today is to lay out most of the proof of Schlessinger's criterion, leaving certain aspects of it to exercises. The bulk of the proof is contained in proving the existence of a hull.

4.1. **Background results.** The following proposition sheds some further light on the structure of deformation functors, and will come up in the proof of Schlessinger's criterion.

Proposition 4.1.1. *Let F be a deformation functor, and $A' \to A$ a small thickening with kernel I. Then for every $\eta \in F(A)$, the set of $\eta' \in F(A')$ restricting to η, when non-empty, has a transitive action of $T_F \otimes_k I$. This action commutes with morphisms of deformation functors $F' \to F$.*

Moreover, the condition (H4) is equivalent to the condition that for all small thickenings and all $\eta \in F(A)$ lifting to A', the above action is free.

In order to check the hull condition, we will also need the following concept:

Definition 4.1.2. A surjection $p : A' \to A$ in $\mathrm{Art}(\Lambda, k)$ is **essential** if for any morphism $q : A'' \to A'$ such that $p \circ q$ is surjective, we have that q is surjective.

Lemma 4.1.3. *If $p : A' \to A$ is a small thickening, then p is not essential if and only if p has a section, which is to say there exists a homomorphism $s : A \to A'$ such that $p \circ s = \mathrm{id}$.*

Both of these results are left as exercises.

4.2. **The existence of a hull.** The following proposition is the hard direction of Schlessinger's theorem.

Proposition 4.2.1. *Suppose that F is a predeformation functor satisfying (H1), (H2), and (H3). Then F has a hull.*

Proof of proposition. The proof breaks up into two parts; constructing the hull, and verifying that it is indeed a hull. We first carry out the construction. Let $\mathfrak{n}$ be the maximal ideal of Λ, and if $r = \dim T_F$, set $S = \Lambda[[t_1, \ldots, t_r]]$, and let $\mathfrak{m}$ be the maximal ideal of S. We will construct a hull R as a quotient of S by a certain ideal J, which we construct inductively. We set $J_2 = \mathfrak{m}^2 + \mathfrak{n}S$, so that $S/J_2 \cong k[T_F^*]$, and let $R_2 = S/J_2$. Choosing an isomorphism $S/J_2 \cong k[T^*F]$ is equivalent to choosing an identification of $t_1, \ldots, t_r$ with a basis for T^*F, which we now do. The dual basis of t_i then give in particular an r-tuple of elements of T_F, which by (H2) (used inductively to describe $F(k[T^*F])$ as the r-fold product of T_F with itself), gives us a $\xi_2 \in F(R_2)$. Following through the definitions, one checks that ξ_2 induces the bijection $T_{R_2} \xrightarrow{\sim} T_F$ dual to the isomorphism we fixed by identifying the t_i as a basis of T_F^*.

We then construct pairs (J_i, ξ_i) inductively for $i > 2$, with $\xi_i \in F(S/J_i)$. Given J_{i-1}, we choose J_i to be the minimal ideal among all J with $\mathfrak{m}J_{i-1} \subseteq J \subseteq J_{i-1}$, and such that ξ_{i-1} can be lifted to some element of $F(S/J)$. We want to see that the collection of such ideals is non-empty and closed under intersection. Non-emptiness is trivial, since J_{i-1} has the desired properties. Similarly, the first condition is trivially closed under arbitrary intersection, so it is enough to check the second. For this, we note that the ideals J correspond to vector subspaces of the finite-dimensional vector space $J_{i-1}/\mathfrak{m}J_{i-1}$, so it is in fact enough to check that the condition is closed under finite, and hence under pairwise intersections. But suppose that J and

J' satisfy the two conditions. Again working in the vector space $J_{i-1}/\mathfrak{m}J_{i-1}$, we see that we can replace J' with some larger J'' satisfying the same conditions, with $J \cap J'' = J \cap J'$, and further $J + J'' = J_{i-1}$. Then we have $S/J \times_{S/J_{i-1}} S/J'' \cong S/(J \cap J'')$, so by (H1) we find we can lift ξ_{i-1} to $S/(J \cap J'')$, and $J \cap J'' = J \cap J'$ satisfies our conditions, as desired, and we can define J_i as the minimal such ideal, and choose ξ_i to be any lift of ξ_{i-1} to S/J_i.

We now take J to be the intersection of all the J_i, and set $R = S/J$. Because $\mathfrak{m}^i \subseteq J_i$ for all i, we also have $R = \varprojlim_i R/J_i$, and we can set $\xi = \varprojlim_i \xi_i$. Note that our construction for $i = 2$ then means that $T_R \cong T_F$, so it remains to check that $h_R \to F$ is smooth.

In order to check smoothness, we suppose we have $p : A' \to A$ a small thickening, and $\eta' \in F(A')$ such that $F(p)(\eta') = \eta \in F(A)$. We also suppose we have $u : R \to A$ such that $\hat{F}(u)(\xi) = \eta$. By the definition of smoothness, we need to show that there exists a morphism $u' : R \to A'$ lifting u, and with $\hat{F}(u')(\xi) = \eta'$. We first claim that there exists a u' with $p \circ u' = u$. Since A is an Artin ring, u factors through R_i for some i, and although this map may not lift, we show that it is possible to produce a lift to a map $R_{i+1} \to A'$ recovering the given map $R_i \to A$, which then induces a lift $R \to A'$. This may be rephrased equivalently as follows: we need to find a morphism $v : R_{i+1} \to R_i \times_A A'$ commuting with the projections to R_i. Now, $p_1 : R_i \times_A A' \to R_i$ is a small thickening because $A' \to A$ is. If p_1 has a section, we can define v simply using the section. On the other hand, if p_1 does not have a section, then p_1 is essential. If we choose any map $S \to A'$ such that the composed map to A agrees with $S \to R \to R_i \to A$, we obtain a map $w : S \to R_i \times_A A'$ making the following square commutative:

$$\begin{array}{ccc} S & \xrightarrow{\;w\;} & R_i \times_A A' \\ \downarrow & & \downarrow{\scriptstyle p_1} \\ R_{i+1} & \longrightarrow & R_i. \end{array}$$

Because p_1 is essential, we see that w is surjective, and in order to produce the desired map v it is enough to show that $\ker w \supseteq J_{i+1}$. But applying (H1) to $R_i \times_A A'$, we have that ξ_i lifts to $R_i \times_A A'$, so by the definition of J_{i+1}, we find $J_{i+1} \subseteq \ker w$, as desired. We have therefore proven that the desired u' exists.

It remains to check that we can modify u' without changing $p \circ u'$ so that $F(u')(\xi) = \eta'$. Thus, we wish to modify u' within $\bar{h}_R(p)^{-1}(\eta)$, and $F(u')(\xi)$ will vary within $F(p)^{-1}(\eta)$. We know from Proposition 4.1.1 that $T_F \otimes I$ acts transitively and functorially on both $F(p)^{-1}(\eta)$ and $\bar{h}_R(p)^{-1}(\eta)$ (using the given isomorphism $T_F \cong T_R$ for the latter), so there is some $\tau \in T_F \otimes I$ sending $F(u')(\xi)$ to η', and by functoriality, the same τ sends u' to some v' with the desired properties. This completes the proof that (R, ξ) is smooth over F, and hence a hull. □

5. Wrapping up Schlessinger's criterion

We begin by completing the proof of Schlessinger's theorem, and then discuss further examples.

5.1. **The rest of the proof.** It is now straightforward to complete the proof of Schlessinger's criterion.

Proof of Schlessinger's criterion. We now know that (H1)-(H3) imply that F has a hull. Conversely, suppose that (R, ξ) is a hull for F. Since $T_R \cong T_F$ and is finite-dimensional because R is Noetherian, we see immediately that F satisfies (H3). Now suppose we have $p' : A' \to A$ and $p'' : A'' \to A$ in $\mathrm{Art}(\Lambda, k)$, where p'' is a surjection, and suppose we have objects $\eta' \in F(A')$ and $\eta'' \in F(A'')$ restricting to the same object $\eta \in F(A)$. Since we saw in the exercises that a smooth morphism is surjective, there exists a $u' : R \to A'$ such that $u'(\xi) = \eta'$, and additionally by smoothness applied to p'', we have $u'' : R \to A''$ with $u''(\xi) = \eta''$ and $p'' \circ u'' = p' \circ u'$. If we set $\zeta = u' \times u''(\xi)$ in $F(A' \times_A A'')$, we then see that ζ projects to η' and η'', which shows that (H1) is satisfied. Now we suppose that $A = k$ and $A'' = k[\epsilon]$, and want to show that the ζ we constructed is unique. But suppose $\omega \in F(A' \times_A A'')$ also restricts to η' and η''. Using the same u', we can use smoothness applied to $A' \times_k k[\epsilon] \to A'$ to find a $v'' : R \to k[\epsilon]'$ such that $u' \times v''(\xi) = \omega$. But because $T_F \cong T_R$ we have $u'' = v''$, so $\omega = \zeta$, as desired.

Now it remains to see that if in addition (H4) is satisfied, our hull in fact prorepresents F, and that conversely if F is prorepresentable, (H4) is satisfied. For the first statement, we suppose (R, ξ) is a hull for F, and (H4) is satisfied, and we argue that $h_R(A) \xrightarrow{\sim} F(A)$ for all $A \in \mathrm{Art}(\Lambda, k)$ by induction on the length of A. Let $p : A' \to A$ a small thickening with kernel I, and suppose $h_R(A) \xrightarrow{\sim} F(A)$. For each $\eta \in F(A)$, by Proposition 4.1.1 and (H4) we have that $h_R(p)^{-1}(\eta)$ and $F(p)^{-1}(\eta)$ are compatibly torsors over $T_F \otimes I$; the surjectivity of $h_R(p)^{-1}(\eta) \to F(p)^{-1}(\eta)$ then implies bijectivity, proving the induction step and allowing us to conclude that (R, ξ) prorepresents F. Finally, for the converse we note that by the fact that our ring fiber products are fiber products in the categories of rings we consider, if F is prorepresentable then we have (1) bijective for all $A' \to A$, $A'' \to A$, and in particular (H4) is satisfied. □

5.2. **Deformations of quotient sheaves.** We have already checked (H1) and (H2) for deformations of schemes, and, in the exercises, deformations of sheaves. Of the three original predeformation functors we defined, that leaves only deformations of closed subschemes, which we now address. In fact, we address a more general situation: that of deformations of quotient sheaves.

Example 5.2.1. Deformations of a quotient sheaf. Let X_Λ be a scheme over $\operatorname{Spec} \Lambda$, with quasicoherent sheaf $\mathcal{E}_\Lambda$. Write X and $\mathcal{E}$ for the restrictions to $\operatorname{Spec} k$. Let $\mathcal{E} \twoheadrightarrow \mathcal{F}$ be a quasicoherent quotient sheaf. The predeformation functor $\mathrm{Def}_{\mathcal{F},\mathcal{E}} : \mathrm{Art}(\Lambda, k) \to \mathrm{Set}$ is defined by sending A to the set of quotient sheaves $\mathcal{F}_A$ of $\mathcal{E}$ which are flat over A, and restrict to $\mathcal{F}$ over $\operatorname{Spec} k$.

Note that as in the case of closed subschemes, there is a notion of equality for quotient sheaves, since quotients correspond to kernel subsheaves. Thus, in this case we do not have to concern ourselves with isomorphisms. In fact, without any further hypotheses, we have that (H1) and (H2) are satisfied by $\mathrm{Def}_{\mathcal{F},\mathcal{E}}$:

Theorem 5.2.2. *With notation as above,* $\mathrm{Def}_{\mathcal{F},\mathcal{E}}$ *is a deformation functor, and indeed always satisfies (H4) as well.*

If further $\mathcal{E}$ *is coherent over* k*, and* X *is proper, then (H3) is satisfied, so* $\mathrm{Def}_{\mathcal{F},\mathcal{E}}$ *is prorepresentable.*

Setting $\mathcal{E}_\Lambda = \mathcal{O}_{X_\Lambda}$, we immediately obtain the following corollary:

Corollary 5.2.3. *Let* X_Λ *be a scheme over* Λ*, with restriction* X *to* k*, and let* $Z \subseteq X$ *be a closed subscheme. Then* $\mathrm{Def}_{Z,X}$ *is a deformation functor, and satisfies (H4).*

If further X *is proper over* k*, then (H3) is satisfied, so* $\mathrm{Def}_{Z,X}$ *is prorepresentable.*

Proof of theorem (sketch). The proof that $\mathrm{Def}_{\mathcal{F},\mathcal{E}}$ is a deformation functor proceeds in much the same way as the proofs for Def_X and $\mathrm{Def}_{\mathcal{E}}$. Indeed, given $A' \to A, A'' \to A$ and quotients $\mathcal{F}_{A'}, \mathcal{F}_{A''}$ restricting to a given $\mathcal{F}_A$ on A, one can define $\mathcal{F}_B$ on $B := A' \times_A A''$ as $\mathcal{F}_{A'} \times_{\mathcal{F}_A} \mathcal{F}_{A''}$, and even though we need not have $\mathcal{E}_\Lambda|_B = \mathcal{E}_\Lambda|_{A'} \times_{\mathcal{E}_\Lambda|_A} \mathcal{E}_\Lambda|_{A''}$, we nonetheless have an induced quotient map $\mathcal{E}_\Lambda|_B \twoheadrightarrow \mathcal{F}_B$, which one can check has the desired properties. This proves (H1), and (H2) and (H4) follow easily because our objects have no non-trivial automorphisms, so one can check directly that the above construction is an inverse to (1).

Finally, to check (H3), we mention that the tangent space is given by $H^0(X, \mathcal{H}om(\mathcal{G}, \mathcal{F}))$, where $\mathcal{G} := \ker(\mathcal{E} \to \mathcal{F})$. This is left as an amusing exercise. This space is finite-dimensional when X is proper and $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is coherent, which is the case under our hypotheses. □

Having reduced deformations of subschemes to deformations of quotient sheaves, we go further with another reduction, allowing us to easily treat another deformation problem.

Example 5.2.4. Deformations of a morphism. Let X_Λ and Y_Λ be schemes over $\operatorname{Spec}\Lambda$. Write X and Y for the restrictions to $\operatorname{Spec} k$. Let $f : X \to Y$ be a morphism over k. The predeformation functor $\operatorname{Def}_f : \operatorname{Art}(\Lambda, k) \to \operatorname{Set}$ is defined by sending A to the set of morphisms $f_A : X_\Lambda|_A \to Y_\Lambda|_A$ over A.

Rather generally, we can use graphs to reduce deformations of morphisms to deformations of subschemes. We have:

Corollary 5.2.5. *Suppose that X_Λ and Y_Λ are locally of finite type over Λ, and that X_Λ is flat and Y_Λ is separated over Λ. Then Def_f is a deformation functor, and satisfies (H4).*

If further X and Y are proper over k, (H3) is also satisfied, so Def_f is prorepresentable.

Proof. We claim that this is a special case of deformations of closed subschemes of $X_\Lambda \times_\Lambda Y_\Lambda$, by considering the graph of f. We note that the flatness of X_Λ implies the flatness of the graph of a deformation of f, while the separatedness of Y_Λ ensures we obtain a closed subscheme. The main point to check is to check that conversely, a flat deformation of a graph is still a graph, which is to say that under our finiteness hypotheses, if we have a morphism from a flat scheme, the condition that it is an isomorphism may be checked on fibers. This is Corollary 17.9.5 of [8]. □

Remark 5.2.6. In fact, in the non-proper case one can define corresponding deformation problems in which one imposes hypotheses of proper support, and this makes it possible to prove prorepresentability for Def_f in the case that only X_Λ is proper.

Remark 5.2.7. We conclude with a remark that these prorepresentability results require fewer hypotheses than global representability results for the corresponding functors – the Quot scheme, Hilbert scheme, and Hom scheme. Indeed, Grothendieck was frustrated that the global functors were not always representable in the proper case, but required projectivity hypotheses.

However, the fact that the local functors are always prorepresentable under the suitable properness hypotheses points to a slight generalization of scheme, the notion of algebraic space developed by D. Knutson and M. Artin. Artin showed that while Hilbert schemes for proper schemes may not be represented by schemes, they are at least represented by algebraic spaces, and in practice this is often just as good as having representability by a scheme.

6. Dimensions of hulls

A key technical result underyling Mori's bend and break argument for the existence of rational curves on Fano varieties involved an argument to use deformation theory to give a lower bound for the dimension of certain moduli spaces of maps. This bound is given in terms of tangent and obstruction spaces.

6.1. **Obstruction theories and the statement.**

We begin with some general comments on obstructions spaces for predeformation functors. Artin's definition of obstruction theory is somewhat complicated, and not quite what we need, so we give a simpler definition:

Definition 6.1.1. A morphism $A' \to A$ in $\operatorname{Art}(\Lambda, k)$ with kernel I is a **thickening** if $I\mathfrak{m}_{A'} = 0$, so that I inherits a k-vector space structure.

Note that this is a specialization to $\operatorname{Art}(\Lambda, k)$ of the ring maps in Martin's definition of a "deformation situation", since the map to k is always surjective.

Definition 6.1.2. Given a predeformation functor F, an **obstruction theory** for F taking values in a k-vector space V consists of the data, for each thickening $A' \to A$ with kernel I, and each $\eta \in F(A)$, of an element $\mathrm{ob}(\eta, A') \in V \otimes_k I$ such that:

(i) $\mathrm{ob}(\eta, A') = 0$ if and only if there exists $\eta' \in F(A)$ with $\eta'|_A = \eta$;
(ii) given an intermediate thickening $A' \to B$ with kernel $J \subseteq I$, we have that $\mathrm{ob}(\eta, B)$ is the image of $\mathrm{ob}(\eta, A')$ under the natural map $V \otimes_k I \to V \otimes_k (I/J)$;

Mori proved the following theorem in the special case of deformations of morphisms. However, the argument generalizes almost immediately.

Theorem 6.1.3. *Let F be a deformation functor satisfies (H3), so that T_F is finite-dimensional, and F has a hull (R, ξ). Suppose also that F has an obstruction theory taking values in a vector space V. Then we have:*

$$\dim \Lambda + \dim_k T_F - \dim_k V \leqslant \dim R \leqslant \dim \Lambda + \dim_k T_F,$$

and if further Λ is regular and the first inequality is an equality, we have that R is a local complete intersection ring.

6.2. **The proof.** The version of this theorem in the usual sources only discusses the case that F is prorepresentable, but the following lemma reduces the general case to this one:

Lemma 6.2.1. *Suppose that F_1, F_2 are functors* $\mathrm{Art}(\Lambda, k) \to \mathrm{Set}$*, and we have a morphism $\pi : F_1 \to F_2$ which is smooth, and an obstruction theory for F_2 taking values in V. Then π induces an obstruction theory for F_1 taking values in V.*

Proof. Given $\eta \in F_1(A)$, and a small extension $A' \to A$, we can define the obstruction ob_η to be simply $\mathrm{ob}_{\pi(\eta)}$. The smoothness of π then implies that η can be lifted to A' if and only if $\pi(\eta)$ can be lifted to A', so the main conditions for an obstruction theory are satisfied, and it remains only to check functoriality, which follows from functoriality of the obstruction theory given for F_2 together with the required functoriality of π. □

Proof of theorem. It is clear that the lemma reduces the theorem down to the case that $F = h_R$, since if R is a hull for F, we have $T_F \cong T_R$ and h_R is smooth over F. We therefore suppose that in fact F is prorepresentable, so that $F = h_R$.

In this case, we work explicitly: if we write $d = \dim T_F$, Schlessinger's construction of R is as a quotient of $S := \Lambda[[t_1, \ldots, t_d]]$ by some ideal J, so to prove the theorem, it is enough to show that the number of generators of J is bounded above by $\dim V$.

By the Artin-Rees lemma, $J \cap \mathfrak{m}_S^n \subset J\mathfrak{m}_S$ for some n. We now set $A' = \Lambda[[t_1, \ldots, t_d]]/(\mathfrak{m}_S J + \mathfrak{m}_S^n)$, and $A = R/\mathfrak{m}_R^n = \Lambda[[t_1, \ldots, t_d]]/(J + \mathfrak{m}_S^n)$, so that we get a small extension

$$0 \to I \to A' \to A \to 0$$

with $I = (J + \mathfrak{m}_{\tilde{R}}^n)/(\mathfrak{m}_{\tilde{R}} J + \mathfrak{m}_{\tilde{R}}^n) = J/\mathfrak{m}_{\tilde{R}} J$. From the natural map $R \to A$ we obtain an object $\xi_A \in F(A)$, with an obstruction $\mathrm{ob}(\xi_A, A') \in V \otimes_k I$ to extending ξ_A to A'. We can write $\mathrm{ob}(\xi_A, A') = \sum_{j=1}^{\dim V} v_j \otimes \bar{x}_j$, where the v_j form a basis for V, and the $\bar{x}_j$ are the images in I of elements $x_j \in J$. We then consider the ring $B := A'/(x_1, \ldots, x_{\dim V})$; this surjects onto A with kernel I', and we again have an obstruction $\mathrm{ob}(\xi_A, B)$ to extending ξ_A to B.

We note that again by the functoriality of obstructions, we will necessarily have $\mathrm{ob}(\xi_A, B) = 0$. Thus, ξ_A may be lifted to B, and because $F = h_R$, this means we can lift the map $R \to A$ to a map $R \to B$. We wish to show that this implies

$$J \subseteq \mathfrak{m}_S J + (\{x_j\}_j) + \mathfrak{m}_S^n. \tag{2}$$

which is equivalent to the stronger assertion that we have a lifting $R \to B$ which commutes with the natural quotient maps from $\Lambda[[t_1, \dots, t_d]]$ to R and to B. Now, if we are given any lifting, we have

$$\begin{array}{ccccc} S = \Lambda[[t_1, \dots, t_d]] & \longrightarrow & R & & \\ \downarrow \varphi & & \downarrow & \searrow & \\ S = \Lambda[[t_1, \dots, t_d]] & \longrightarrow & B & \longrightarrow & A, \end{array}$$

and we can fill in the dashed arrow φ to make the diagram commute by choosing appropriate values for $\varphi(t_i)$, $i = 1, \dots, d$. By hypothesis, φ commutes with the maps to A, so must be the identity modulo $J + \mathfrak{m}_S^n$. In particular, we conclude that φ induces the identity map on $\mathfrak{m}_S/\mathfrak{m}_S^2$, so is an isomorphism, and in particular maps $\mathfrak{m}_S$ bijectively to itself. Since φ factors the given map $S \to B$, we have $\varphi^{-1}(J) \subset J + \mathfrak{m}_S^n$, so that $J \subset \varphi(J) + \varphi(\mathfrak{m}_S^n) = \varphi(J) + \mathfrak{m}_S^n$. But by commutativity of the maps to R and B, we see $\varphi(J) \subseteq \mathfrak{m}_S J + (\{x_j\}_j) + \mathfrak{m}_S^n$, and putting these together gives (2).

Since we had originally $J \cap \mathfrak{m}_S^n \subseteq \mathfrak{m}_S J$, we finally conclude that J is contained in, hence equal to $\mathfrak{m}_S J + (\{x_j\}_j)$. By Nakayama's lemma, we conclude that J is generated by $\{x_j\}_j$, as desired. □

We have the following silly corollary, which can of course be proved directly:

Corollary 6.2.2. *If $R \in \widehat{\mathrm{Art}}(\Lambda, k)$ is smooth over Λ, then $R \cong \Lambda[[t_1, \dots, t_r]]$ for some r.*

Remark 6.2.3. In his guest lecture, Jason discussed a somewhat more informative proof of the above theorem. The proof is too involved to give all the details, but we recap the main points. Given a ring $R \in \widehat{\mathrm{Art}}(\Lambda, k)$, let $r = \dim T_R$, and $S = \Lambda[[t_1, \dots, t_r]]$, and consider R as a quotient of S with kernel ideal I. Let $V = \mathrm{Hom}_R(I/I^2, k)/\mathrm{Hom}_S(\hat{\Omega}, k)$, where $\hat{\Omega}$ is the free module on S generated by the dt_i, and the map sends a functional $(dt_i \mapsto c_i)$ to $(f \mapsto \sum_{i=1}^r \frac{\partial f}{\partial t_i} c_i)$. Then V is an s-dimensional vector space, where s is the minimal number of generators of I/I^2, and the first point is that $\bar{h}_R$ has a canonical obstruction theory taking values in V.

The next point is that for any obstruction theory for $\bar{h}_R$ taking values in some V', there is a unique linear map $V \to V'$ inducing the obstruction theory from the canonical one, and moreover this map is linear. Since V is s-dimensional, this implies that the number of generators for I is bounded from above by the dimension of any obstruction theory, which recovers the theorem above.

6.3. **Examples.** One can often express tangent and obstruction space dimensions in terms of cohomology groups, and this is a powerful tool for computing dimensions. We mention two examples where the computation can be made particularly simply.

Example 6.3.1. The context that Mori applied his theorem was to spaces of morphisms from a curve to a variety. For simplicity, we assume that X, Y are smooth over k, with X a proper curve, and $f : X \to Y$ is given. Then deformations of f have tangent space given by $H^0(X, f^*T_Y)$, and an obstruction theory with values in $H^1(X, f^*T_Y)$, so we find that to obtain a lower bound on the dimension at f of the space of morphisms $X \to Y$, we only need to compute the Euler characteristic $\chi(f^*T_Y)$, which can be computed using Riemann-Roch from the dimension Y and the degree on X of f^*T_Y.

Example 6.3.2. Another case in which one can often reduce the computation to Riemann-Roch is for abstract deformations of a smooth, proper surface X. Here, we know the tangent space is given by $H^1(X, T_X)$, and obstructions lie in $H^2(X, T_X)$, and we want to compute $h^1(X, T_X) - h^2(X, T_X)$ to obtain a lower bound. This is reduced to a Riemann-Roch computation when we know $H^0(X, T_X)$, which is the dimension of the space of infinitesimal automorphisms of X, and in characteristic 0 is equal to the dimension of the automorphism group of X. For instance, if X has finitely many automorphisms (which is the case in particular if it is of general type), we have $H^0(X, T_X) = 0$, and then $h^1(X, T_X) - h^2(X, T_X) = -\chi(T_X)$.

7. Effectivization and algebraization

We have now concluded our discussion of functors of Artin rings, but certain important questions remain. Two such questions are the following:

Question. (Effectivity) Suppose F is a deformation functor obtained from a global moduli problem, and R a complete local Noetherian ring, and we have an object $\eta \in \hat{F}(R)$. When does η actually correspond to an object of the original moduli problem over $\operatorname{Spec} R$?

Question. (Algebraization) In the above situation, when does an object over $\operatorname{Spec} R$ arise as the base change of an "algebraic" object – that is, an object over a ring of finite type over the base?

Neither property is always satisfied, although both are satisfied in many important cases. At first glance, it may appear that effectivity is a minor technical issue, and algebraization is more substantive. However, it turns out that effectivity is more frequently a sticking point.

We note that an important special case of both questions is when (R, η) is actually a hull for F; these questions then ask for the existence of a universal object over $\operatorname{Spec} R$, and over an algebraic subring, respectively.

7.1. Effectivity.

Grothendieck proved the first major effectivity result, the Grothendieck existence theorem for coherent sheaves which was discussed in the background lectures. We recall a rough form of its statement:

Theorem 7.1.1. *Let $f : X \to \operatorname{Spec} A$ be a proper morphism, with A a complete local Noetherian ring. Let $A_n = A/\mathfrak{m}_A^n$, and $X_n = X \otimes_A A_n$. Suppose $\mathcal{F}_n$ is a compatible collection of coherent sheaves on X_n. Then there exists a coherent sheaf $\mathcal{F}$ on X restricting to $\mathcal{F}_n$ on each X_n.*

One can apply Grothendieck's theorem immediately to moduli of coherent sheaves on a proper scheme, concluding that formal objects (i.e., objects of $\hat{F}(R)$) can always be effectivized. However, this does not extend to other important examples, most notably moduli of schemes.

Example 7.1.2. A smooth projective surface S is a **K3 surface** if $\omega_S \cong \mathcal{O}_S$, and $H^1(S, \mathcal{O}_S) = 0$. It turns out that if we fix a K3 surface S, and consider deformations of S, the hull is 20-dimensional, but roughly speaking only 19 of these dimensions can be effectivized. This is a consequence of the fact that the space of K3 surfaces considered as complex manifolds is 20-dimensional, but the space of algebraic K3 surfaces is a countable union of 19-dimensional closed subspaces. It is possible to deform formally in the direction of an analytic, non-algebraic surface, but such a deformation cannot be effectivized.

Nonetheless, all is not lost. The typical patch is to consider moduli of polarized varieties – that is, varieties together with a choice of ample line bundle. This is similar to, but weaker than, restricting our consideration to subvarieties of projective space, but it turns out to rigidify the problem sufficiently to allow effectivization to work. As one might expect, the proof involves a reduction to Grothendieck's existence theorem for coherent sheaves, although we note that the necessity of working with morphisms of sheaves means that the proof requires a stronger statement of the existence theorem than the version given above, asserting that one actually obtains an appropriate categorical equivalence.

7.2. **Algebraization.** Having more or less dealt with the question of effectivity, we now turn to algebraization. We should not expect any completely general results, but Artin showed that if we restrict our attention to deformations (R, ξ) which induce smooth maps $\bar{h}_R \to F$ (in which case we will say (R, ξ) is smooth over F), it is possible to prove general results under surprisingly mild hypotheses. We need one preliminary definition:

Definition 7.2.1. Let $F : \mathrm{Sch}_S \to \mathrm{Set}$ be a contravariant functor. We say F is **locally of finite presentation** over S if for all filtering projective systems of affine schemes $Z_\lambda \in \mathrm{Sch}_S$, we have

$$\varinjlim F(Z_\lambda) = F(\varprojlim Z_\lambda).$$

Note that the arrows here are in the opposite direction from the effectivity condition; here, we consider direct limits of rings. The reason for this definition is that it is shown in Proposition 8.14.2 of [7] that if $F = h_X$ for some $X \in \mathrm{Sch}_S$, the above definition is in fact equivalent to the usual definition of the morphism $X \to S$ being locally of finite presentation.

Artin proved his algebraization theorem using his earlier results on approximations, and we are therefore required to restrict our base scheme S to be locally of finite type over a field or an excellent Dedekind domain. Given $F : \mathrm{Sch}_S \to \mathrm{Set}$ and $\eta_0 \in F(\mathrm{Spec}\, k)$, we denote by $\bar{F}_{\eta_0}$ the predeformation functor obtained by setting $\bar{F}_{\eta_0}(A) = \{\eta \in F(A) : \eta|_k = \eta_0\}$. Artin's theorem is then the following:

Theorem 7.2.2. *Suppose* $F : \mathrm{Sch}_S \to \mathrm{Set}$ *is locally of finite presentation, and* $\eta_0 \in F(\mathrm{Spec}\, k)$ *for a field* k*, where* $\mathrm{Spec}\, k \to S$ *is given and of finite type, having image* $s \in S$*. Let* R *be a complete local Noetherian* $\mathcal{O}_{S,s}$*-algebra with residue field* k*, and* $\xi \in F(\mathrm{Spec}\, R)$ *inducing* η_0 *on* k *and smooth over* $\bar{F}_{\eta_0}$*. Then there is an* S*-scheme* X *of finite type, with a closed point* $x \in X$ *having residue field* k*, and an element* $\eta \in F(X)$ *such that there exists an isomorphism* $\hat{\mathcal{O}}_{X,x} \xrightarrow{\sim} R$ *with* η *inducing* $\xi_n \in F(R/\mathfrak{m}_R^{n+1})$ *for each* n*.*

Note that effectivity is built into the hypotheses, since we start with an element of $F(\mathrm{Spec}\, R)$, and not merely an inverse system of elements over the Artin quotients of R. Also note that it is not necessarily the case that η induces $\xi \in F(R)$, unless (as is often the case), the element ξ is uniquely determined by the truncations ξ_n. In this case, we have the following uniqueness theorem, which says that the formal deformation theory is determining X uniquely, at least etale locally.

Theorem 7.2.3. *Under the hypotheses of the above theorem, suppose further that* ξ *is uniquely determined by the* ξ_n*. Then* (X, x, η) *is unique up to etale base change, in the sense that any two such triples are related by a common etale morphism.*

These theorems are Theorems 1.6 and 1.7 of [1], respectively.

8. Groupoid perspective

As we've seen in Max's and Martin's lectures, when working with moduli problems and related deformations it is frequently beneficial to work in the context of categories fibered in groupoids. We've even seen this indirectly in the first couple days, when defining some of the basic examples of predeformation functors. For instance, we noted that Def_X can't be defined formally from a global functors of flat families of schemes, since the functor doesn't keep track of the data of pullback morphisms. On the other hand, when working with categories fibered in groupoids, one can formally restrict from global to local moduli problems, and still obtain good behavior.

However, it turns out that thinking in terms of categories fibered in groupoids can also shed additional light on Schlessinger's criterion, and leads to other good behavior as well.

8.1. **Deformation stacks.** We first observe that (H1) and (H2) can be subsumed by a single somewhat stronger condition in the groupoid setting, which we make precise, following some background definitions.

Definition 8.1.1. Let C be a category. A category $\mathcal{S}$ **cofibered in groupoids** over C is a category $\mathcal{S}$ together with a functor (usually omitted from the notation) to C which makes $\mathcal{S}$ into a category fibered in groupoids over C^{opp}.

Definition 8.1.2. Let D be a category such that every morphism is an isomorphism. We say D is the **trivial groupoid** if every pair of objects has a unique morphism between them.

We frequently refer to "the" trivial groupoid, since any trivial groupoid is equivalent to the category with one object and only the identity morphism.

For categories cofibered in groupoids over $\text{Art}(\Lambda, k)$, the condition corresponding to (H1) and (H2) has gone by various names: Rim called a category satisfying this condition a "homogeneous groupoid", while Artin referred to the condition itself as $S1'$. We propose the following terminology.

Definition 8.1.3. A category $\mathcal{S}$ cofibered in groupoids over $\text{Art}(\Lambda, k)$ is a **deformation stack** if $\mathcal{S}_k$ is the trivial groupoid, and if for every pair of morphisms $A' \to A, A'' \to A$ in $\text{Art}(\Lambda, k)$, with $A'' \to A$ a surjection, the following conditions are satisfied:

(i) for any pair of objects $\eta_1, \eta_2 \in \mathcal{S}_{A' \times_A A''}$, the natural map
$$\text{Mor}_{A' \times_A A''}(\eta_1, \eta_2) \to \text{Mor}_{A'}(\eta_1|_{A'}, \eta_2|_{A'}) \times_{\text{Mor}_A(\eta_1|_A, \eta_2|_A)} \text{Mor}_{A''}(\eta_1|_{A''}, \eta_2|_{A''})$$
is a bijection;

(ii) given a pair of objects $\eta' \in \mathcal{S}_{A'}$ and $\eta'' \in \mathcal{S}_{A''}$, and an isomorphism $\varphi : \eta'|_A \to \eta''|_A$, there exists an object $\zeta \in \mathcal{S}_{A' \times_A A''}$ restricting to η' and η'' on A' and A'', and recovering φ on A.

To any deformation stack $\mathcal{S}$, we have the associated functor to sets $F_{\mathcal{S}}$ obtained by sending A to the set of isomorphism classes of $\mathcal{S}_A$. We observe the following:

Proposition 8.1.4. *Let $\mathcal{S}$ be a deformation stack. Then $F_{\mathcal{S}}$ is a deformation functor.*

Proof. That $F_{\mathcal{S}}$ is a predeformation functor follows from the hypothesis that $\mathcal{S}_k$ is the trivial groupoid. We thus need to see that (H1) and (H2) follow from the additional conditions. But (H2) follows immediately from condition (ii) of a deformation stack, while (H1) (and more generally, that (1) is a bijection whenever $A = k$) then follows from condition (i), since any two objects have a unique isomorphism over k, so any isomorphisms over A' and A'' necessarily come from an isomorphism over $A' \times_k A''$. □

8.2. **Ubiquity and utility of deformation stacks.** On a purely formal level, it seems that the conditions for a deformation stack are stronger than a deformation functor. However, in practice it seems that any time one has an argument showing that a given functor satisfies Schlessinger's (H1) and (H2), then the same argument will show that actually the deformation stack condition is satisfied as well. In particular, we see that this is the case for deformations of abstract schemes, and deformations of quasicoherent sheaves on a scheme: Proposition 3.2.2 immediately implies that Def_X (given the natural groupoid structure) is a deformation stack, and one can similarly deduce that $\text{Def}_{\mathcal{E}}$ is a deformation stack from Lemma 3.2.1. In another direction, in Lemma 1.4.4 of [4] Martin gives an argument showing that any deformation problem obtained from a point of an Artin stack will satisfy the conditions of a deformation stack.

We can therefore think of the deformation stack condition as "explaining" where (H1) and (H2) come from, and why it is natural for these conditions to be satisfied. Additionally, Martin's lemma gives an indication of why it is natural to assume that $A'' \to A$ is surjective: there is some natural asymmetry arising from the use of a smooth cover of the stack by a scheme, and the need to apply the formal criterion of smoothness on one side. Indeed, the deformation stack axiom is automatically satisfied for a deformation problem coming from a point of a scheme, simply by applying the universal property of fiber products. In order to generalize to the case of Artin stacks, one then wants to reduce to the case of schemes via the smooth cover, and this is where the formal criterion for smoothness arises.

However, deformation stacks have additional nice behavior building on some of the structure we have discussed for deformation functors. For instance, with a deformation functor F, we had that for any thickening $A' \to A$ with kernel I, and any $\eta \in F(A)$, the vector space $T_F \otimes_k I$ acts transitively on $\{\eta' \in F(A') : \eta'|_A = \eta\}$, but we do not in general obtain a torsor. However, with deformation stacks we can describe a torsor, and we can do similarly with automorphisms:

Proposition 8.2.1. *Suppose $\mathcal{S}$ is a deformation stack, and $A' \to A$ is a thickening with kernel I.*

Then for any $\eta \in \mathcal{S}_A$, we have that

$$\{(\eta', \varphi) : \eta' \in F(A'), \varphi : \eta'|_A \xrightarrow{\sim} A\}$$

is a pseudotorsor for $T_{\mathcal{S}} \otimes_k I$.

Similarly, if we write ζ_ϵ for the trivial deformation over $k[\epsilon]$ obtained by base change under $k \to k[\epsilon]$, then for any $\eta' \in \mathcal{S}_{A'}$ and any $\varphi \in \mathrm{Aut}(\eta'|_A)$, we have that

$$\{\varphi' \in \mathrm{Aut}(\eta') : \varphi'|_A = \varphi\}$$

is a pseudotorsor for $\mathrm{Aut}(\zeta_\epsilon) \otimes_k I$.

Here, we can put a k-vector space structure on $\mathrm{Aut}(\zeta_\epsilon)$ using the same construction that we used to put a vector space structure on $T_{\mathcal{S}} := T_{F_{\mathcal{S}}}$. An amusing fact is that the additional law we obtain in this way agrees with the composition law on $\mathrm{Aut}(\zeta_\epsilon)$ which is given to us *a priori*; in particular, the deformation stack axioms automatically imply that this automorphism group is abelian.

As pointed out informally by Schlessinger, and formalized by Rim, we also obtain a geometric interpretation for (H4) in the deformation stack context:

Proposition 8.2.2. *Let $\mathcal{S}$ be a deformation stack. Then $F_{\mathcal{S}}$ satisfies (H4) if and only if for every surjection $A' \to A$ in $\mathrm{Art}(\Lambda, k)$, and every $\eta' \in \mathcal{S}_{A'}$, the natural map $\mathrm{Aut}(\eta') \to \mathrm{Aut}(\eta'|_A)$ is a surjection.*

Proof. This is not hard to see, and essentially the same as the proof of the same statement for Def_X; we sketch one direction. If the map on automorphism groups is surjective, and we have $\eta_1, \eta_2 \in \mathcal{S}_{A' \times_A A'}$ which are isomorphic after restriction to the first and second factors, the two isomorphisms differ on A by an element of $\mathrm{Aut}(\eta_1|_A)$, which by hypothesis can be lifted to an element of $\mathrm{Aut}(p_{2*}\eta_1)$. If we modify the isomorphism on the second factor by this automorphism, we obtain isomorphisms on both factors which agree on A, so by the deformation stack axioms we obtain an isomorphism $\eta_1 \to \eta_2$ on all of $A' \times_A A'$. □

When the deformation stack is obtained from an object η_0 of a global stack, the proposition is saying that (H4) is equivalent to the associated Isom stack being smooth at the identity section in $\mathrm{Aut}(\eta_0)$.

8.3. **Deformation stacks as stacks.** We conclude with a brief remark on the analogy between deformation stacks and stacks in the usual sense. From a descent theory perspective, the fiber product rings appearing in Schlessinger's criterion are slightly mysterious, but the basic observation is the following:

Lemma 8.3.1. *We have a correspondence between the fiber product squares considered by Schlessinger*

$$\begin{array}{ccc} A & \longleftarrow & A' \\ \twoheaduparrow & & \uparrow \\ A'' & \longleftarrow & A' \times_A A'' \end{array}$$

and tensor product squares

$$\begin{array}{ccc} B' \otimes_B B'' & \longleftarrow & B' \\ \uparrow & & \twoheaduparrow q' \\ B'' & \underset{q''}{\longleftarrow} & B \end{array}$$

where we further require that $B \hookrightarrow B' \times B''$ and $q''(\ker q')$ be an ideal inside B''. Here the correspondence is given simply by $B' = A'$, $B'' = A''$, and $B = A' \times_A A''$.

Note that the condition that $B \hookrightarrow B' \times B''$ is equivalent to saying that the map

$$\operatorname{Spec} B' \coprod \operatorname{Spec} B'' \to \operatorname{Spec} B$$

be scheme-theoretically surjective, which means it is reasonable to think of $\operatorname{Spec} B'$ and $\operatorname{Spec} B''$ together forming a cover of $\operatorname{Spec} B$, with the intersection then given by $\operatorname{Spec}(B' \otimes_B B'') = \operatorname{Spec} B' \times_{\operatorname{Spec} B} \operatorname{Spec} B''$. Thus, the conditions on isomorphisms and objects in the definition of a deformation stack can really be viewed as analogous to the usual stack conditions, but where we consider a particular strange collection of covers coming from pairs of maps $\operatorname{Spec} B' \to \operatorname{Spec} B$ and $\operatorname{Spec} B'' \to \operatorname{Spec} B$ with asymmetric conditions.

Although these covers are strange from the point of view of "classical" Grothendieck topologies such as the etale topology, more recent topologies such as those introduced by Nisnevich and Voevodsky are strikingly similar, also being defined in terms covers given by pairs of maps with asymmetric conditions on them. In these cases, one considers the smallest Grothendieck topology containing all the given pairs, but in our case, it is not difficult to see that one cannot hope to do so, because the squares we consider are not preserved under base change, so the Grothendieck topology they generate is too large to hope for any descent conditions to hold. The fundamental sticking point is that scheme-theoretic surjectivity is very poorly behaved under base change. We thus see that deformation stacks may be thought of as categories fibered in groupoids which satisfy a stack condition relative to some collection of covers, but it does not appear that the condition can actually be expressed relative to any Grothendieck topology.

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