ON THE RANK OF A TROPICAL MATRIX

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Abstract. This is a foundational paper in tropical linear algebra, which is linear algebra over the min-plus semiring. We compare three natural definitions of the rank of a matrix, called the Barvinok rank, the Kapranov rank and the tropical rank, and we show that they differ in general. Connections to polyhedral geometry, particularly to subdivisions of products of simplices, are emphasized.

1. Introduction

The rank of a matrix M is one of the most important notions in linear algebra. This number can be defined in many different ways. In particular, the following definitions are equivalent:

Definition 1. *The* rank *of M is the largest number r such that M has a nonsingular $r \times r$ minor.*

Definition 2. *The* rank *of M is the dimension of the linear span of its rows (or columns).*

Definition 3. *The* rank *of M is the dimension of the smallest linear space containing the rows (or columns) of M.*

Definition 4. *The* rank *of M is the smallest r for which M can be written as the sum of r rank-1 matrices. A matrix has rank 1 if its rows (or columns) are scalar multiples of each other.*

Our objective is to examine these familiar definitions over an algebraic structure which has no additive inverses. We work over the *tropical semiring* $(\mathbb{R}, \oplus, \odot)$ whose arithmetic operations are

$$a \oplus b \ := \ \min(a,b) \qquad \text{and} \qquad a \odot b \ := \ a + b.$$

The set $\mathbb{R}^n$ of real n-vectors and the set $\mathbb{R}^{m\times n}$ of real $m \times n$-matrices are semimodules over the semiring $(\mathbb{R}, \oplus, \odot)$. The operations of matrix addition and matrix multiplication are well defined.

All four definitions of rank make sense over the tropical semiring $(\mathbb{R}, \oplus, \odot)$. For Definition 4 we say that an $m \times n$-matrix M has *rank one* if it is the tropical product of an $m \times 1$-matrix and a $1 \times n$-matrix. For Definition 1 we say that an $n \times n$-matrix $M = (m_{ij})$ is *tropically singular* if it lies in the tropical hypersurface defined by the determinant. What this means is that the minimum in

$$\det(M) \quad = \quad \bigoplus_{\sigma \in S_n} m_{1\sigma_1} \odot m_{2\sigma_2} \odot \cdots \odot m_{n\sigma_n} \quad = \quad \min\{\, m_{1\sigma_1} + m_{2\sigma_2} + \cdots + m_{n\sigma_n} \, : \, \sigma \in S_n \,\}$$

is attained at least twice [11].

As in ordinary linear algebra, every $m \times n$-matrix M defines a tropically linear map $\mathbb{R}^m \to \mathbb{R}^n$. The image of M is a polyhedral complex in $\mathbb{R}^n$. Following [7], we identify this polyhedral complex with its image in the tropical projective space $\mathbb{TP}^{n-1} = \mathbb{R}^n/\mathbb{R}(1,1,\ldots,1)$, which is known as the *tropical convex hull* of (the rows of) M. Equivalently, this *tropical polytope* is the set consisting of

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all tropical linear combinations of the rows of M. The number referred to in Definition 2 is the dimension of this tropical polytope plus one.

The discrepancy between Definition 2 and Definition 3 comes from the crucial distinction between tropical polytopes and tropical linear spaces, as explained in [11, §1]. The latter are described in [13] where it is shown that they are parametrized by the tropical Grassmannian.

Our aim in this paper is to compare the various notions of rank in tropical linear algebra. In Section 2 we study Definition 4, which is known as the *Barvinok rank*. The notion of Barvinok rank naturally arises in the context of combinatorial optimization [3, 5].

In Section 3 we study Definition 3, which we call the *Kapranov rank*. This notion is most natural from the point of view of algebraic geometry, where tropical arithmetic arises as the "tropicalization" of arithmetic in a power series ring. It has good algebraic and geometric properties but is difficult to characterize combinatorially; for instance, it depends on the base field of the power series ring, which here we take to be the complex numbers $\mathbb{C}$, unless otherwise stated.

In Section 4 we prove that Definitions 1 and 2 are equivalent, and we call this the *tropical rank*. This notion of tropical rank is the best one from a geometric and combinatorial perspective. For instance, the tropical convex hulls this definition is dependent on correspond to regular subdivisions of products of simplices [7]. In Section 5, we show that tropical rank of a matrix can also be characterized in terms of combinatorial conditions on the corresponding subdivision.

The main result of this paper relates the three notions of rank:

Theorem 5. *For every $m \times n$-matrix M with entries in the tropical semiring $(\mathbb{R}, \oplus, \odot)$, we have*

$$\textit{tropical rank}(M) \quad \leq \quad \textit{Kapranov rank}(M) \quad \leq \quad \textit{Barvinok rank}(M). \tag{1}$$

Both of these inequalities can be strict.

There is a heuristic reason why the tropical rank is always the smallest of the three: in any sensible definition of rank of a point configuration, we would expect the rank not to increase upon adding linear combinations of points. Since adding all the linear combinations of points yields a set of dimension equal to the tropical rank, we would certainly hope that any definition of rank we could come up with would be greater than or equal to the tropical rank. Why the Kapranov rank is smaller than the Barvinok rank is more subtle. It is proved in Proposition 17, and we will see in Example 36 that this inequality is false when $\mathbb{C}$ is replaced by a finite field.

The second inequality is strict for many matrices (see Proposition 11 for an instructive example), but it requires some effort to find matrices for which the first inequality is strict. Such matrices will be constructed in Section 7 by relating Kapranov rank to realizability of matroids. In Sections 5 and 6 we study two special cases where the tropical and Kapranov rank coincide for an $m \times n$-matrix: when any of them is at most two or equal to $\min(m, n)$. In the case of rank two, we interpret the set of $m \times n$-matrices enjoying this property as the space of trees with m leaves and n marked points. The equality when the Kapranov rank equals $\min(m, n)$ can be derived from the result, by Bernstein and Zelevinsky [4], that the maximal minors of a matrix of unknowns form a universal Gröbner basis; we offer a direct combinatorial proof in Section 7.

Summing up, the three definitions of rank studied in this paper generally disagree, and they have different flavors (combinatorial, algebraic, geometric). But they all share some of the familiar properties of matrix rank over a field. Among them: the rank of a matrix and its transpose are the same; the rank of a minor cannot exceed that of the whole matrix; the rank is invariant under (tropical) multiplication of rows or columns by constants, and under insertion of a row or column obtained as a combination of others; the rank is subadditive (rank of $M \oplus N$ is at most the sum of the ranks of M and N); and the rank of $M \odot N$ is at most the maximum of the ranks of M and N.

2. The Barvinok rank

Barvinok, Johnson and Woeginger [3] showed that, for fixed r, the Traveling Salesman Problem can be solved in polynomial time if the distance matrix is the tropical sum of r matrices of tropical rank one (with $\oplus$ as "max" instead of "min"). The following definition is consistent with [5].

Definition 6. *The* Barvinok rank *of an $m \times n$-matrix M is the smallest integer r for which there exist vectors $X_1, \dots, X_r \in \mathbb{R}^m$ and vectors $Y_1, \dots, Y_r \in \mathbb{R}^n$ such that*

$$M \quad = \quad X_1^T \odot Y_1 \;\oplus\; X_2^T \odot Y_2 \;\oplus\; \cdots \;\oplus\; X_r^T \odot Y_r. \tag{2}$$

Equivalently, M is the tropical sum of r matrices each of which has tropical rank one.

For example, the following equation shows a 3×3-matrix which has Barvinok rank two:

$$\begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 4 & 5 & 4 \end{pmatrix} \quad = \quad (0,2,4)^T \odot (0,1,2) \quad \oplus \quad (0,0,4)^T \odot (2,1,0).$$

This matrix also has tropical rank 2 and Kapranov rank 2 because the matrix is tropically singular. The row vectors lie on the tropical line in $\mathbb{TP}^2 = \mathbb{R}^3/\mathbb{R}(1,1,1)$ defined by $20 \odot x_1 \oplus 0 \odot x_2 \oplus 1 \odot x_3$.

We next present two reformulations of the definition of Barvinok rank: in terms of tropical convexity [7], and via a "tropical morphism" in matrix space. Recall from [7] that the *tropical convex hull* of m vectors in $\mathbb{TP}^{n-1}$ is the image (under left multiplication) of the $m \times n$-matrix whose rows are the given vectors. In this manner, the image of M becomes a *tropical polytope.*

Proposition 7. *Let M be a real $m \times n$-matrix. The following properties are equivalent:*

(a) *M has Barvinok rank at most r.*
(b) *The rows of M lie in the tropical convex hull of r points in $\mathbb{TP}^{n-1}$.*
(c) *There are matrices $X \in \mathbb{R}^{m \times r}$ and $Y \in \mathbb{R}^{r \times n}$ such that $M = X \odot Y$. Equivalently, M lies in the image of the following tropical morphism, which is defined by matrix multiplication:*

$$\phi_r \;:\; \mathbb{R}^{m \times r} \times \mathbb{R}^{r \times n} \;\rightarrow\; \mathbb{R}^{m \times n} \;,\;\; (X, Y) \;\mapsto\; X \odot Y. \tag{3}$$

Proof. Let $M_1, \dots, M_m \in \mathbb{R}^n$ be the row vectors of M. Let $X_1, \dots, X_r \in \mathbb{R}^m$ and $Y_1, \dots, Y_r \in \mathbb{R}^n$ be the rows of two unspecified matrices $X \in \mathbb{R}^{r \times m}$ and $Y \in \mathbb{R}^{r \times n}$. Let X_{ki} denote the ith coordinate of X_k. The following three algebraic identities are easily seen to be equivalent:

(a) $M \;=\; X_1^T \odot Y_1 \;\oplus\; X_2^T \odot Y_2 \;\oplus\; \cdots \;\oplus\; X_r^T \odot Y_r$,
(b) $M_i \;=\; X_{1i} \odot Y_1 \;\oplus\; X_{2i} \odot Y_2 \;\oplus\; \cdots \;\oplus\; X_{ri} \odot Y_r$ for all $i = 1, \dots, m$, and
(c) $M = X^T \odot Y$.

Statement (b) says that each row vector of M lies in the tropical convex hull of $Y_1, \dots, Y_r$. The entries of the matrix X are the multipliers in that tropical convex combination. This shows that the three conditions (a), (b) and (c) in the statement of the proposition are equivalent. ☐

Part (b) of Proposition 7 suggests that the Barvinok rank of a tropical matrix is more an analogue of the non-negative rank of a matrix than of the usual rank. Recall (e.g. from [6]) that the *non-negative rank* of a real non-negative matrix $M \in \mathbb{R}^{m \times n}$ is the smallest r for which M can be written as a product of an $m \times r$ and an $r \times n$ non-negative matrices. Equivalently, it is the smallest r for which the rows (or columns) of M lie in the positive hull of r non-negative vectors. Compare this with the definition of Barvinok rank. Equation (2) can be rephrased as "M is a tropical linear combination of r matrices of rank 1". But in tropical geometry, linear combinations do not define a "linear span". Rather, they define a "positive span" or "convex hull" [7, 11]. For more information on non-negative rank see [6] and for the connection to rank over other semigroup rings see [9].

We next take a closer look at the structure of the tropical matrix multiplication map ϕ_r.

Proposition 8. *The map ϕ_r is piecewise-linear. The domains of linearity form a fan in $\mathbb{R}^{m\times r} \times \mathbb{R}^{r\times n}$. This fan is the common refinement of the normal fans of mn simplices of dimension $(r-1)$.*

Proof. Let $U = (u_{ij})$ and $V = (v_{jk})$ be matrices of indeterminates of format $m \times r$ and $r \times n$ respectively. The entries of the (classical – not tropical) matrix product UV are the mn quadratic polynomials $u_{i1}v_{1k} + u_{i2}v_{2k} + \cdots + u_{ir}v_{rk}$. The Newton polytope of each such quadric is an $(r-1)$-dimensional simplex P_{ij}. Let $P = \sum_{i=1}^{m}\sum_{j=1}^{n} P_{ij}$ denote the Minkowski sum of these mn simplices. This is a polytope of dimension $\bigl(2\cdot\min(m,n)-1\bigr)(r-1)$ sitting inside $\mathbb{R}^{m\times r}\times\mathbb{R}^{r\times n}$.

The ik-coordinate of the tropical map ϕ_r takes a pair of matrices (X, Y) to the real number $\min(x_{i1}+y_{1k}, x_{i2}+y_{2k}, \cdots, x_{ir}+y_{rk})$. This function is the support function of the simplex P_{ij}. It is linear on each cone in the normal fan of P_{ij}. Hence ϕ_r is a linear map on the common refinement of the normal fans of the simplices P_{ij}. This common refinement is the normal fan of their Minkowski sum P. We conclude that ϕ_r is piecewise-linear on the normal fan of P. □

Corollary 9. *If $r = 2$ then the tropical morphism ϕ_2 is piecewise-linear with respect to an arrangement of mn hyperplanes in $\mathbb{R}^{m\times 2}\times\mathbb{R}^{2\times n}$.*

Proof. If $r = 2$ then each P_{ij} is a line segment, and their Minkowski sum P is a zonotope of dimension $2\cdot\min(m,n)-1$. The normal fan of the zonotope P is a hyperplane arrangement, and it follows from the previous proof that ϕ_r is piecewise linear on that hyperplane arrangement. □

Example 10. Let $m = n = 3$ and $r = 2$. Then P is a four-dimensional zonotope with nine zones in $\mathbb{R}^{12} = \mathbb{R}^{3\times 2}\times\mathbb{R}^{2\times 3}$. This zonotope has 230 vertices, so the dual hyperplane arrangement has 230 maximal regions. The tropical morphism ϕ_2 maps each of these 230 regions linearly onto an 8-dimensional cone in $\mathbb{R}^{3\times 3}$. The image of ϕ_2 is the set of all tropically singular 3×3-matrices. This is a polyhedral fan with 15 maximal cones. It is the codimension one skeleton of the normal fan of the four-dimensional Birkhoff polytope (the convex hull of all six 3×3-permutation matrices). □

By Proposition 7, the set of all Barvinok matrices of rank $\leq r$ is image of the tropical morphism ϕ_r. In particular, this set is a polyhedral fan in $\mathbb{R}^{m\times n}$, as in the previous example. The distinction between the Barvinok rank and the Kapranov rank, to be discussed in the next section, can be explained by the following general fact of tropical algebraic geometry: For most polynomial maps, *the image of the tropicalization is strictly contained in the tropicalization of the image.*

We next present an example of a matrix which will show that the Barvinok rank can be much larger than the other two notions of rank. The matrix to be considered is the $n\times n$-matrix

$$C_n \quad = \quad \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{4}$$

This looks like the unit matrix (in classical arithmetic) but it is far from being a unit matrix in tropical arithmetic, where 0 is the neutral element for $\odot$ and ∞ is the neutral element for $\oplus$. The Barvinok rank of this matrix was computed in [5]:

Proposition 11. *The Barvinok rank of the matrix C_n is the smallest integer r such that*

$$n \leq \binom{r}{\lfloor \frac{r}{2} \rfloor}.$$

Proof. Let r be an integer and assume that $n \leq \binom{r}{\lfloor r/2\rfloor}$. We first show that Barvinok rank $(C_n) \leq r$. Let $S_1, \ldots, S_n$ be distinct subsets of $\{1, \ldots, r\}$ each having cardinality $\lfloor r/2\rfloor$. For each $k \in 1, \ldots, r$, we define an $n \times n$-matrix $X_k = (x_{ij}^k)$ with entries in $\{0,1,2\}$ as follows:

$$x_{ij}^k = 0 \ \text{ if } \ k \in S_i \backslash S_j \,, \quad x_{ij}^k = 2 \ \text{ if } \ k \in S_j \backslash S_i, \ \text{ and } \quad x_{ij}^k = 1 \ \text{ otherwise.}$$

The matrix X_k has tropical rank one. To see this, let $V_k \in \{0,1\}^n$ denote the vector with ith coordinate equal to one or zero depending on whether k is an element of S_i or not. Then we have

$$X_k \quad = \quad V_k^T \odot (\, 1 \odot (-V_k)\,).$$

To prove Barvinok rank$\,(C_n) \leq r$, it now suffices to establish the identity

$$C_n \quad = \quad X_1 \oplus X_2 \oplus \cdots \oplus X_r$$

Indeed, all diagonal entries of the matrices on the right hand side are 1, and the off-diagonal entries (for $i \neq j$) of the right hand side are $\min(x_{ij}^1, x_{ij}^2, \ldots, x_{ij}^r) = 0$, because $S_i \backslash S_j$ is non-empty.

To prove the converse direction, we consider an arbitrary representation

$$C_n \quad = \quad Y_1 \oplus Y_2 \oplus \cdots \oplus Y_r$$

where the matrices $Y_k = (y_{ij}^k)$ have tropical rank one. For each k we set $T_k := \{(i,j) : y_{ij}^k = 0\}$. Since the matrices Y_k are non-negative and have tropical rank one, it follows that each T_k is a product $I_k \times J_k$, where I_k and J_k are subsets of $\{1, \ldots, n\}$. Moreover, we have $I_k \cap J_k = \emptyset$ because the diagonal entries of Y_k are not zero. For each $i = 1, \ldots, n$ we set

$$S_i := \{k : i \in I_k\} \quad \subseteq \quad \{1, \ldots, r\}.$$

We claim that no two of the sets $S_1, \ldots, S_n$ are contained in one another. Sperner's Theorem [1] then proves that $n \leq \binom{r}{\lfloor r/2 \rfloor}$. To prove the claim, observe that if $S_i \subset S_j$ then the entry $y_{i,j}^k$ cannot be zero for any k. Indeed, if $k \in S_i \subseteq S_j$ then $j \in I_k$ implies $j \notin J_k$. And if $k \notin S_i$ then $i \notin I_k$. $\square$

For example, C_6 has Barvinok rank 4, as the following decomposition shows:

$$C_6 = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 2 & 2 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 & 2 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 2 & 2 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 & 1 \end{pmatrix}.$$

Similarly, C_{36} has Barvinok rank 8, even though all its proper 35×35 minors have Barvinok rank 7 (and its 8×8 minors have Barvinok rank at most 5). Asymptotically,

$$\text{Barvinok rank}\,(C_n) \quad \sim \quad \log_2 n.$$

We will see in Examples 16 and 21 that the Kapranov rank and tropical rank of C_n are both two.

To finish, let us mention some algorithmic results from [5]: the computation of the Barvinok rank of a matrix is an NP-complete problem, but determining whether a matrix has Barvinok rank 2 can be done in time $O(mn)$. Matrices of Barvinok rank 2 are characterized as being *Monge matrices*, with perhaps permuted rows and columns. It is also shown that a matrix has Barvinok rank 2 if and only if all its 3×3 minors do, a result of which we give a different proof in Section 6.

3. The Kapranov rank

The tropical semiring has a strong connection to power series rings and their algebraic geometry. We review the basic setup from [13, 14]. Let $K = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series with complex coefficients. The elements in K are formal power series $f = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$, where $a_1 < a_2 < \cdots$ are rational numbers that have a common denominator. Let $\deg : K^* \to \mathbb{Q}$ be the natural valuation sending a non-zero Puiseux series f to its degree a_1. For any two elements $f, g \in K$, we have $\deg(fg) = \deg(f) + \deg(g) = \deg(f) \odot \deg(g)$. In general we also have $\deg(f+g) = \min(\deg(f), \deg(g)) = \deg(f) \oplus \deg(g)$, unless there is a cancellation of leading terms. Thus the tropical arithmetic is naturally induced from ordinary arithmetic in power series fields.

The field $K = \mathbb{C}\{\{t\}\}$ is algebraically closed of characteristic zero. If I is any ideal in $K[x_1, \ldots, x_n]$ then we write $V(I)$ for its variety in the n-dimensional algebraic torus $(K^*)^n$. Thus the elements of $V(I)$ are vectors $x(t) = (x_1(t), \ldots, x_n(t))$ where each $x_i(t)$ is a Puiseux series and $f(x(t)) = 0$ for each polynomial $f \in I$. It is sometimes better to enlarge the field K and to allow all formal power series $f = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$ where the a_i can be real numbers, not just rationals. We denote this larger field by $\tilde{K}$ and we write $\tilde{V}(I)$ for the variety in $(\tilde{K}^*)^n$ defined by I. The degree map can be applied coordinatewise, giving rise to a map which takes vectors of power series into $\mathbb{R}^n$:

$$\deg : (\tilde{K}^*)^n \;\to\; \mathbb{R}^n, \;(f_1(t), \ldots, f_n(t)) \;\mapsto\; \big(\deg(f_1), \ldots, \deg(f_n)\big).$$

We define the *tropical variety* of I, denoted $\mathcal{T}(I) \subset \mathbb{R}^n$, to be the image of $\tilde{V}(I)$ under the map deg. In [13, 14], $\mathcal{T}(I)$ is described as the topological closure of the image of $V(I)$ under deg, but this is equivalent, and the following alternative description of the tropical variety is given:

Theorem 12. *The tropical variety $\mathcal{T}(I)$ is the set of vectors $w \in \mathbb{R}^n$ such that the initial ideal $\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle$ contains no monomial. The dimension of $\mathcal{T}(I)$ is the dimension of $V(I)$.*

This theorem is basically due to Misha Kapranov, who proved it in an unpublished manuscript for the case when I is a principal ideal. The extension to arbitrary ideals was first stated in [14] and fully proved in [13]. The initial ideal $\mathrm{in}_w(I)$ can be computed from any generating set of I by computing a Gröbner basis with respect to any term order that refines w. This shows that $\mathcal{T}(I)$ is a polyhedral subcomplex of the Gröbner fan of I, and it leads to an algorithm for computing $\mathcal{T}(I)$.

We are now ready to state the tropical version of Definition 3 in precise terms. The name "Kapranov rank" was chosen to give credit to Misha's pivotal role in starting tropical geometry.

A *tropical linear space* in $\mathbb{R}^n$ is any subset $\mathcal{T}(I)$ where I is an ideal generated by affine-linear forms $a_1x_1 + \cdots + a_nx_n + b$ in $\tilde{K}[x] = \tilde{K}[x_1, \ldots, x_n]$. Note that here the scalars $a_1, \ldots, a_n, b$ are power series in t with complex coefficients, the choice of the complex numbers being a critical one. If I is the principal ideal generated by one affine-linear form $a_1x_1 + \cdots + a_nx_n + b$, then $\mathcal{T}(I)$ is a *tropical hypersurface.* Tropical linear spaces were studied in [13], where it was shown that they are parametrized by the *tropical Grassmannian.* Every tropical linear space L is a finite intersection of tropical hyperplanes, but not conversely, and the number of of tropical hyperplanes needed is generally larger than the codimension of L.

Definition 13. *The* Kapranov rank *of a matrix $M \subset \mathbb{R}^{m\times n}$ is the smallest dimension of any tropical linear space containing the rows of M.*

It is not completely apparent in this definition that the Kapranov rank of a matrix and its transpose are the same, but it follows from our next result. Let J_r denote the ideal generated by all the $(r+1)\times(r+1)$-subdeterminants of an $m \times n$-matrix of indeterminates (x_{ij}). This is a prime ideal of dimension $rm + rn - r^2$, and the generating determinants form a Gröbner basis. The variety $V(J_r)$ consists of all $m \times n$-matrices with entries in K^* whose (classical) rank is at most r.

Theorem 14. *For a real matrix $M = (m_{ij}) \in \mathbb{R}^{m\times n}$ the following statements are equivalent:*

(a) *The Kapranov rank of M is at most r.*
(b) *The matrix M lies in the tropical determinantal variety $\mathcal{T}(J_r)$.*
(c) *There exists an $m \times n$-matrix $F = \big(f_{ij}(t)\big)$ with non-zero entries in the field $\tilde{K}$ such that the rank of F is less than or equal to r and $\deg(f_{ij}) = m_{ij}$ for all i and j.*

The power series matrix F in part (c) is called a *lift* of M. We abbreviate this as $\deg(F) = M$.

Proof. The equivalence of (b) and (c) is simply our definition of tropical variety applied to the ideal J_r, as over the field $\tilde{K}$ lying in the variety of the determinantal ideal J_r is equivalent to having rank at most r. To see that (c) implies (a), consider the linear subspace of $\tilde{K}^n$ spanned by the rows of F. This is an r-dimensional linear space over a field, so it is defined by an ideal I generated by $n-r$ linearly independent linear forms in $\tilde{K}[x_1,\dots,x_n]$. The tropical linear space $\mathcal{T}(I)$ contains all the row vectors of $M = \deg(F)$.

Conversely, suppose that (a) holds, and let L be a tropical linear space of dimension r containing the rows of M. Pick a linear ideal I in $\tilde{K}[x_1,\dots,x_n]$ such that $L = \mathcal{T}(I)$. By applying the definition of tropical variety to the ideal I, we see that each row vector of M has a preimage in $\tilde{V}(I) \subset (\tilde{K}^*)^n$ under the degree map. Let F be the $m\times n$-matrix over $\tilde{K}$ whose rows are these preimages. Then the row space of F is contained in the variety defined by I, so we have $\operatorname{rank}(F) \le r$, and $\deg(F) = M$ as desired. ☐

Corollary 15. *The Kapranov rank of a matrix $M \in \mathbb{R}^{m\times n}$ is the smallest rank of any lift of M.*

For the case of Kapranov rank one, the ideal J_1 is generated by the 2×2-minors $x_{ij}x_{kl} - x_{il}x_{kj}$ of the $m\times n$-matrix (x_{ij}). Therefore, any matrix of Kapranov rank one must certainly satisfy the corresponding linear equations $m_{ij} + m_{kl} = m_{il} + m_{kj}$. This happens if and only if there exist real vectors $X = (x_1,\dots,x_m)$ and $Y = (y_1,\dots,y_n)$ with

$$m_{ij} \;=\; x_i + y_j \text{ for all } i,j \quad\iff\quad m_{ij} \;=\; x_i \odot y_j \text{ for all } i,j \quad\iff\quad M \;=\; X^T \odot Y.$$

If this happens, we can lift M to a matrix of rank one simply by substituting $t^{m_{ij}}$ for m_{ij}. Therefore, a matrix M has Kapranov rank one if and only if it has Barvinok rank one. In general, the Kapranov rank can be much smaller than the Barvinok rank, as the following example shows.

Example 16. Let $n \ge 3$ and consider the matrix C_n in Proposition 11 of Section 2. The matrix C_n does not have Kapranov rank one, so it has Kapranov rank at least two. Let $a_3, a_4, \dots, a_n$ be distinct nonzero complex numbers. Consider the matrix

$$F_n \quad=\quad \begin{pmatrix} t & 1 & t-a_3 & t-a_4 & \cdots & t-a_n \\ 1 & t & 1 & 1 & \cdots & 1 \\ t+a_3 & 1+a_3t & t & t-a_4+a_3 & \cdots & t-a_n+a_3 \\ t+a_4 & 1+a_4t & t-a_3+a_4 & t & \cdots & t-a_n+a_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t+a_n & 1+a_nt & t-a_3+a_n & t-a_4+a_n & \cdots & t \end{pmatrix}.$$

The matrix F has rank 2 because the i-th row (for $i \ge 3$) equals the first row plus a_i times the second row. Since $\deg(F_n) = C_n$, we conclude that C_n has Kapranov rank two.

The two-dimensional tropical plane containing the rows of C_n is the two-dimensional fan L in $\mathbb{R}^n$ which consists of the n cones $\{x_i \ge x_1 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_n\}$; this is the tropical variety defined by the ideal in $\mathbb{C}[x_1,\dots,x_n]$ generated by $n-2$ linear forms with generic coefficients in $\mathbb{C}$. Its image in $\mathbb{TP}^{n-1}$ is the line all of whose tropical Plücker coordinates are zero [13].

The following proposition establishes half of Theorem 5.

Proposition 17. *Every matrix* $M \in \mathbb{R}^{m\times n}$ *satisfies Kapranov rank*$(M) \leq$ *Barvinok rank*(M)*, and this inequality can be strict.*

Proof. Suppose that M has Barvinok rank r and write $M = M_1 \oplus \cdots \oplus M_r$ where each M_i has Barvinok rank one. Then M_i has Kapranov rank one, so there exists a rank one matrix F_i over $\tilde{K}$ such that $\deg(F_i) = M_i$. Moreover, by multiplying the matrices F_i by generic complex numbers if necessary, we can choose F_i such that there is no cancellation of leading terms in t when we form the matrix $F = F_1 + \cdots + F_r$. This means $\deg(F) = M$. Clearly, the matrix F has rank $\leq r$. Theorem 14 then implies that M has Kapranov rank $\leq r$. Example 16 shows that the inequality can be strict. □

A general algorithm for computing the Kapranov rank of a matrix M involves computing a Gröbner basis of the determinantal ideal J_r. Suppose we wish to decide whether a given real $m \times n$-matrix $M = (m_{ij})$ has Kapranov rank $> r$. To decide this question, we fix any term order $\prec_M$ on the polynomial ring $\mathbb{C}[x_{ij}]$ which refines the partial ordering on monomials given assigning weight m_{ij} to the variable x_{ij}, and we compute the reduced Gröbner basis $\mathcal{G}$ of J_r in the term order $\prec_M$. For each polynomial g in $\mathcal{G}$, we consider its leading form $\mathrm{in}_M(g)$ with respect to the partial ordering coming from M. As noted in [14, §1], we have $\mathrm{in}_{\prec_M}\big(\mathrm{in}_M(g)\big) = \mathrm{in}_{\prec_M}(g)$ for all $g \in \mathcal{G}$.

The ideal generated by the set of leading forms $\big\{\mathrm{in}_M(g) : g \in \mathcal{G}\big\}$ is the initial ideal $\mathrm{in}_M(J_r)$. Let x^{all} denote the product of all mn unknowns x_{ij}. The second step in our algorithm is to compute the saturation of the initial ideal with respect to the coordinate hyperplanes:

$$\big(\mathrm{in}_M(J_r) \;:\; \langle x^{all}\rangle^{\infty}\big) \quad = \quad \big\{\, f \in \mathbb{C}[x_{ij}] \;:\; f\cdot (x^{all})^s \,\in\, J_r \text{ for some } s \in \mathbb{N}\big\}. \tag{5}$$

Computing such an ideal quotient, given the generators $\mathrm{in}_M(g)$, is a standard operation in computational commutative algebra. It is a built-in command (called `saturate`) in software systems such as `CoCoA`, `Macaulay 2` or `Singular`. The following is a direct consequence of Theorems 12 and 14. It also corrects an incorrect statement about universal Gröbner bases made in [13, §1].

Corollary 18. *The matrix* M *has Kapranov rank* $> r$ *if and only if (5) is the unit ideal* $\langle 1\rangle$*.*

In this section, we have defined Kapranov rank in terms of power series arithmetic over the complex field $\mathbb{C}$, which is a canonical choice for doing algebraic geometry. However, the same definition works over any field F. One can consider the Puiseux series field $K = F\{\{t\}\}$ with either rational or real exponents. Note the the former is not algebraically closed if F is algebraically closed of characteristic p, but this need not concern us. We denote the latter by $\tilde{K}$ as before. All we need is the degree map $(\tilde{K}^*)^n \to \mathbb{R}^n$. We define a *tropical linear space over the field* F to be the image under "deg" of any linear subspace of the $\tilde{K}$-vector space $\tilde{K}^n$, and we define *the Kapranov rank over* F using Definition 13. Theorem 14 is true over all fields, but Proposition 17 is true only over infinite fields because in its proof we needed to take generic coefficients, which makes no sense over a finite field. Indeed, Example 36 in Section 6 shows a matrix whose Kapranov rank over the 2-element field $\mathbb{F}_2$ is greater than the Barvinok rank. Even over infinite fields, the Kapranov rank of a matrix may differ from field to field; we will discuss this further and give examples in Section 7.

4. THE TROPICAL RANK

In tropical geometry, the space of linear combinations of a set of vectors behaves more like a convex (or positive) hull than a linear subspace. Indeed, it always has a 1-dimensional lineality space spanned by the vector $(1, \ldots, 1)$, but upon quotienting out by this 1-dimensional space, the resulting object is bounded. This quotient is the tropical analogue to the construction of projective space $\mathbb{P}^{n-1}$ from $\mathbb{C}^{n-1}$. It sends the object into *tropical projective space* $\mathbb{TP}^{n-1}$. Following [7], we call the set of tropical linear combinations of a finite set of vectors their *tropical convex hull.*

Definition 19. *Let $M \subset \mathbb{R}^{m \times n}$ be a matrix. Then the* tropical rank *of M is the dimension of the tropical convex hull of the row vectors of M as a subset of $\mathbb{R}^n$; in other words, it is the dimension of the set $\left\{\bigoplus_{i=1}^m c_i \odot v_i \,:\, c_1, \ldots, c_m \in \mathbb{R}\right\} \subset \mathbb{R}^n$, where $v_1, \ldots, v_m$ are the row vectors of M.*

It is shown in [7] that the dimension of the tropical convex hull of the columns of a matrix is equal to the dimension of the tropical convex hull of its rows, as in ordinary geometry. That is to say, the tropical ranks of a matrix and its transpose are the same.

Let $V = \{v_1, \ldots, v_m\} \subseteq \mathbb{R}^n$ be a finite set of vectors. For each sequence $S = (S_1, \ldots, S_n)$ of subsets $S_i \subseteq \{1, \ldots, m\}$, we denote by X_S the region of $\mathbb{TP}^{n-1}$ defined by the inequalities $x_k - x_j \leq v_{ik} - v_{ij}$ for all $k \in \{1, \ldots, n\}$ and all i, j such that $i \in S_j$. One of the main results in [7] is that the tropical convex hull of V equals the union of the X_S which are bounded. The sequence S is called the *type* of a given point $x \in X_S$ in the tropical polytope. Recalling that x is a componentwise minimum of various $c_i v_i$, this is essentially tantamount to saying that the j-th coordinate of x comes from the i-th term of this sum. The dimension of a particular X_S can be easily computed from the combinatorics of the set S: let G_S be the graph G_S which has vertex set $1, \ldots, n$, with i and j connected by an edge if $S_i \cap S_j$ is nonempty. The dimension of X_S is one less than the number of connected components of G_S [7].

Recall that a square matrix M is *tropically nonsingular* if the tropical determinant, as defined in the introduction, does not vanish on M. That is to say, if the sum $\sum_{i=1}^n M_{i\sigma(i)}$ achieves its minimum exactly once as σ ranges over all permutations in the symmetric group. This algebraic concept is equivalent to the definition of tropical rank:

Theorem 20. *Let M be a matrix. Then the tropical rank of M is equal to the largest r for which M has a tropically nonsingular $r \times r$ submatrix.*

Proof. Let $V = \{v_1, \ldots, v_m\}$ be the rows of M, and let $P \subset \mathbb{TP}^{n-1}$ be the tropical convex hull of V. First, suppose M has a nonsingular r by r submatrix N; we want to show that P has dimension at least $r - 1$. By projecting onto the r coordinates corresponding to the columns of N and looking at the subpolytope consisting of the convex hull of the r points corresponding to the rows of N, we can assume that M is itself a r by r matrix which is tropically nonsingular. Also, without loss of generality, we can assume that the minimum over $\sigma \in S_r$ of

$$f(\sigma) \quad = \quad \sum_{i=1}^{r} v_{i\sigma(i)} \tag{6}$$

is uniquely achieved when σ is the identity element $e \in S_r$.

We now claim that the region $X_{(1,\ldots,r)}$ in M is of dimension $r - 1$. The inequalities defining this region are precisely $\{x_k - x_j \leq v_{jk} - v_{jj}\}$ for $j \neq k$. These inequalities define a full-dimensional polyhedron if and only if no linear combination of them is equal to the inequality $0 \leq c$ for c non-positive. However, such a linear combination is easily seen to imply that some other $\sigma \in S_r$ has $f(\sigma) \leq f(e)$, a contradiction.

For the converse, suppose that P has dimension $r - 1$; we want to show the existence of a nonsingular r by r submatrix of M. Pick a region X_S of dimension $r - 1$. The graph G_S has r connected components, so in particular we can pick r elements of $\{1, \ldots, m\}$ of which no two appear in a common S_j. In particular, each i in this set appears in some $S_{z(i)}$ not containing any other element from this set.

Assume without loss of generality that this set of elements is $\{1, \ldots, r\}$, and that $z(i) = i$. We claim that the tropical submatrix consisting of the first r rows and columns of M is tropically nonsingular. Indeed, we have:

$$f(\sigma) - f(e) = \sum_{i=1}^{r} v_{i\sigma(i)} - \sum_{i=1}^{r} v_{ii} \tag{7}$$

$$= \sum_{i=1}^{r} \big(v_{i\sigma(i)} - v_{ii}\big) \tag{8}$$

but whenever $\sigma(i) \neq i$, $v_{i\sigma(i)} - v_{ii} > 0$ since $i \in S_i$ and $\sigma(i) \notin S_i$. Therefore, if σ is not the identity, we have $f(\sigma) - f(e) > 0$, and e is the unique permutation in S_r minimizing the expression (6). So M is indeed tropically nonsingular. ☐

Theorem 20 gives us a straightforward way to evaluate tropical rank, as well as a witness for it.

Example 21. *The tropical rank of the matrix C_n in Proposition 11 equals two, since all its 3×3 minors are tropically singular, while the principal 2×2 minors are not.*

We can now finish the proof of Theorem 5.

Proposition 22. *Every matrix $M \in \mathbb{R}^{m\times n}$ satisfies tropical rank$(M) \leq$ Kapranov rank(M), and this inequality can be strict.*

Proof. If M has a tropically nonsingular $r \times r$ minor, then any lift of M must have the lift of this $r \times r$-minor be nonsingular, since the leading exponent of its determinant occurs only once in the sum. Consequently, no lift of M can have rank less than r, so the Kapranov rank of M must be at least r by Theorem 14. That the inequality can be strict will be shown in Section 7. ☐

We finish this section with a combinatorial description of the tropical rank of a zero-one matrix. The same result applies to any matrix which has only two distinct entries. Let M be an $m \times n$-matrix. The *support* of a row v_i of M is its set of zero entries. We define the *support poset of M* to be the poset of all unions of supports of rows of M.

Proposition 23. *The tropical rank of a zero-one matrix with no row of all ones equals the maximum length of a chain in its support poset.*

The assumption that there is no row of all ones is needed for the statement to hold because a row of zeroes and a row of ones represent the same point, but add two to the length of any chain. The proposition applies to any zero-one matrix by first changing all rows of ones to be rows of zeroes.

Proof. Observe that there is no loss of generality in assuming that every union of supports of rows of M is actually the support of a row. Indeed, the tropical sum of a set of rows gives a row whose support is the union of supports. Hence, if there is a chain with r elements in the support poset we may assume that there is a set of r rows with supports contained in one another. Since there is no row of ones, from this we can easily extract an $r \times r$ minor with zeroes on and below the diagonal and 1's above the diagonal, which is tropically non-singular.

Reciprocally, suppose there is a tropically non-singular $r \times r$ minor N. We claim that the support poset of N has a chain of length r, from which it follows that the support poset of M also has a chain of length r. Assume without loss of generality that the unique minimum permutation sum is obtained in the diagonal. This minimum sum cannot be more than one, because if n_{ii} and n_{jj} are both 1 then changing them for n_{ij} and n_{ji} does not increase the sum. If the minimum is zero, orienting an edge from i to j if entry ij of N is zero yields an acyclic digraph, which admits an ordering. Rearranging the rows and columns according to this ordering yields a matrix with 1's below the diagonal and 0's on the diagonal. The tropical sum of the last i rows (which corresponds to union of the corresponding supports) then produces a vector with 0's exactly in the last i positions. Hence, there is a proper chain of supports of length r.

If the minimum permutation sum in N is 1, then let n_{ii} be the unique diagonal entry equal to 1. The i-th column in N must consist of all 1's: if n_{ji} is zero, then changing n_{ji} and n_{ii} for n_{ij} and n_{jj} does not increase the sum. Changing this column of ones to a column of zeroes does not affect the support poset of N (it just adds an element to every support), and yields a non-singular 0/1 matrix with minimum sum zero to which we can apply the previous argument. □

As an example, let us look again at the matrix C_n with 1's on the diagonal. The supports of its rows are all the sets of cardinality $n-1$ and the support poset consists of them and the whole set $\{1,\dots,n\}$. The maximal chains in the poset have length two.

5. Mixed Subdivisions and Corank One

A useful tool in tropical convexity is the relation of tropical convex hulls to mixed subdivisions of several copies of a simplex. We recall the definition of mixed subdivision, adapted to the case of interest to us. See [12] for more details.

Definition 24. *Let $\Delta^{n-1} \subset \mathbb{R}^{n-1}$ denote the simplex of dimension $n-1$, with vertex set $A = \{e_1,\dots,e_n\}$. Let $m\Delta^{n-1}$ denote its dilation by a factor of m, which we regard as the convex hull of the Minkowski sum $A + A + \cdots + A$. Let $M = (v_{ij}) \subset \mathbb{R}^{m\times n}$ be a matrix.*

For each $i = 1,\dots,m$, let P_i, $i = 1,\dots,m$ be the lifted simplex

$$P_i := \operatorname{conv}(\{(e_1, v_{i1}),\dots,(e_n, v_{in})\} \subset \mathbb{R}^n$$

The regular mixed subdivision *of $m\Delta^{n-1}$ induced by M is the set of projections of the lower faces of the Minkowski sum $P_1 + \cdots + P_m$. Here, a face is called lower if its outer normal cone contains a vector with last coordinate negative.*

It was shown in [7] that there is a bijective correspondence between the types (as defined above) that appear in the convex hull of the rows of M and interior cells in the regular subdivision of a product of simplices induced by M. Via the so-called Cayley trick [12], the latter biject to interior cells in the regular mixed subdivision defined above. Here we provide a short direct proof of the composition of these two bijections:

Lemma 25. *Let $M \subset \mathbb{R}^{m\times n}$ and let $S = (S_1,\dots,S_n)$, where each S_j is a subset of $\{1,\dots,m\}$. Then, the following properties are equivalent:*

1. *The tropical convex hull of the rows of M contains a cell of type S.*
2. *There is a non-negative matrix M' such that M' is obtained from M by adding constants to rows or columns of M, and such that $M'_{ij} = 0$ precisely when $i \in S_j$.*
3. *The regular mixed subdivision of $m\Delta^{n-1}$ induced by M has as a cell the Minkowski sum $\tau_1 + \cdots + \tau_n$ where $\tau_i = \operatorname{conv}(\{e_j : i \in S_j\})$.*

Moreover, if this happens the cells referred to in parts (1) and (3) have complementary dimensions.

Proof. Observe that adding a constant to a column of M amounts to a translation of the corresponding tropical (projective) point set, while adding a constant to a row leaves the point set unchanged. For a given cell in the tropical convex hull, let x be any point in the relative interior of it and let M' be the (unique) matrix obtained by translating the point set by a vector $-x$ and normalizing every point so that its minimum coordinate equals 0. Reciprocally, for a given M' consider the point x whose coordinates are the amounts added to the columns of M to obtain M'. If we take a point x with type exactly S, then by definition it is easy to see that the modified matrix M' has zeroes precisely in entries ij with $i \in S_j$, proving the equivalence of (1) and (2).

For the equivalence of (2) and (3), observe that adding a constant to a row or column of M does not affect the regular mixed subdivision: adding to the i-th row only translates vertically the polytope P_i (and hence the Minkowski sum $\sum P_i$) while adding to a column is an affine transformation that preserves "verticality", hence does not change the lower envelope of $\sum P_i$. Moreover, for a non-negative matrix M' with at least a zero in every column the positions of the zero entries define the face of $\sum P_i$ in the negative vertical direction. Reciprocally, for every cell of the regular mixed subdivision, we can apply a linear transformation to the height function to give that cell height zero and all other vertices positive height (this is what it means to be in the lower envelope.) This height function is precisely the matrix M' in (2), completing the proof of the equivalence of (2) and (3). The assertion on dimensions is easy to prove. ☐

This lemma has two immediate corollaries.

Corollary 26. *Given a matrix M, the poset of types in the tropical convex hull of its rows and the poset of interior cells (cells not contained in the boundary) of the corresponding regular mixed subdivision are antiisomorphic. In other words, the tropical convex hull and the (interior of) the regular mixed subdivision are dual polyhedral complexes.*

Corollary 27. *Let $M \subset \mathbb{R}^{m\times n}$. The tropical rank of M equals n minus the minimal dimension of an interior cell in the regular mixed subdivision of $m\Delta^{n-1}$ induced by M.*

We can use these tools to prove that the tropical and Kapranov ranks of a matrix coincide if the latter is maximal:

Theorem 28. *If an $m \times n$ matrix M has Kapranov rank n then it has tropical rank n, too.*

Proof. Let P be the tropical convex hull of the rows of M, and let S be the regular mixed subdivision of $m\Delta^{n-1}$ associated with M. By Corollary 27, M has tropical rank n if and only if S has an interior vertex. The theorem then follows from the following two lemmas. ☐

Lemma 29. *A matrix M has Kapranov rank smaller than n if and only if the corresponding regular mixed subdivision S has a cell that intersects all facets of $m\Delta^{n-1}$.*

Proof. If M has Kapranov rank smaller than n, then its row vectors are contained in a tropical hyperplane. Since all tropical hyperplanes are translates of one another, there is no loss of generality in assuming that it is the hyperplane $x_1 \oplus \cdots \oplus x_n = 0$. That is, after normalization, all rows of M are non-negative and have at least two zeroes. Then, by Lemma 25, the zero entries of M define a cell $B \in S$ in the none of whose Minkowski summands are single vertices. In particular, for every facet F of Δ^{m-1} and for every $i \in \{1, \ldots, m\}$, the i-th summand of B is at least an edge and hence it intersects F. Hence, B intersects all facets of $m\Delta^{n-1}$.

Reciprocally, if S has a cell B which intersects all facets of $m\Delta^{n-1}$, then there is no loss of generality in assuming that M gives height zero to the entries in that cell and positive to all the others. The condition that B intersects all facets translates to all rows having at least two zeroes. ☐

Observe that the cell in the preceding statement need not be unique. For example, if a tetrahedron is sliced by planes parallel to two opposite edges, then all the (maximal) cells of S meet all the facets.

Lemma 30. *In every polyhedral subdivision of a simplex without interior vertices there is a cell that intersects all of the facets.*

Here, when we say "polyhedral subdivision" we are not a priori fixing the set of vertices to be used.

Proof. Observe that there is no loss of generality in assuming that the polyhedral subdivision S is a triangulation. For a triangulation, we use Sperner's Lemma [1]: "if the vertices of a triangulation of Δ are labeled so that (1) the vertices of Δ receive different labels and (2) the vertices in any face F of Δ receive labels among those of the vertices of F, then there is a fully labeled simplex".

Our goal is to give our triangulation a Sperner labeling with the property that every vertex labeled i lies in the i-th facet of the simplex. The way to obtain this is: the vertex opposite to facet i is labeled $i+1$. More generally, the label i of a vertex v is taken so that v is contained in facet i but not on facet $i-1$. All labels are modulo d. ☐

6. Matrices of rank two

Throughout this section let M be a matrix of tropical rank equal to two, and let P be the tropical convex hull of its rows. By definition of tropical rank, P is one-dimensional. Since it is contractible, it is a tree. Another way of showing this is via the corresponding regular mixed subdivision. Tropical rank 2 means that all the interior cells have codimension zero or one. Hence, the subdivision is made out by slicing the simplex via a certain number of hyperplanes (which do not meet inside the simplex) and its dual graph is a tree.

For the case of Barvinok rank two, we have the following proposition. For matrices of higher Barvinok rank, its natural analogue fails, as Proposition 11 demonstrates.

Proposition 31. *The following are equivalent for a matrix M:*

1. *It has Barvinok rank 2.*
2. *All its 3×3 minors have Barvinok rank 2.*
3. *The tropical convex hull of its rows is a path.*

Proof. (1)⇒(2) is trivial (the Barvinok rank of a minor cannot exceed that of the whole matrix) and (3)⇒(1) is easy: if a tropical polytope is a path, then it is the tropical convex hull of its two endpoints. Proposition 7 then implies that the Barvinok rank is two.

For (2)⇒(3) first observe that the case where M is 3×3 again follows from Proposition 7. We next consider the case where M is $3 \times n$: since the tropical convex hulls of rows and of columns of a matrix are homeomorphic [7], assume that the tropical convex hull of the columns of M is not a path. Then, there are three columns whose tropical convex hull is not a path, and their 3×3 minor has Barvinok rank 3. Finally, if M is of arbitrary size $m \times n$ and the tropical convex hull of its rows is not a path, consider three rows whose tropical convex hull is not a path and apply the previous case to them. ☐

Our goal in the rest of this section is to show that if M has tropical rank 2 then it has Kapranov rank 2, by providing an explicit way of lifting it with rank 2.

Lemma 32. *Let x be a point in the tropical convex hull of the rows of M. Let M' be the matrix obtained by adding $-x$ to every row and then normalizing rows to have zero as their minimal entry. After perhaps reordering the rows and columns, M' has the following block structure:*

$$M' := \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A_k \end{pmatrix},$$

where the matrices A_i have all entries positive and every 2×2 minor has the property that the minimum of its four entries is achieved twice. Each $\mathbf{0}$ represents a matrix of zeroes of the appropriate dimension.

Moreover, the tropical convex hull of the rows of M' is the union of the tropical convex hulls of the row vectors of the blocks augmented by the zero vector (where a block consists of the rows *meeting one of the A_i's) and two of these convex hulls meet only at the point* $\mathbf{0}$*, which is a vertex in all of them.*

Proof. The block decomposition of our matrix can be rephrased as saying that any two given rows have either equal or disjoint supports, where the support of a row is the set of positions where it does not have a zero. To prove that this holds, just observe that if it didn't then we would have the following non-singular minor, where + denotes a strictly positive entry.

$$\begin{pmatrix} 0 & + & + \\ 0 & 0 & + \\ 0 & 0 & 0 \end{pmatrix}$$

The assertion of the 2×2 minors follows from the fact that the non-negative matrix

$$\begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{pmatrix}$$

is tropically singular if and only if the minimum of a, b, c and d is achieved twice.

Finally, the assertion about the convex hulls is trivial, since any linear combination of row vectors from a given block will have all zero entries except in the coordinates corresponding to that block. Any path joining two such points from different blocks will pass through the origin. □

To show how to glue liftings of different blocks, we first describe how to obtain appropriate liftings:

Lemma 33. *Let A be a non-negative matrix and let $\tilde{A}$ be the matrix*

$$\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix}$$

obtained by adjoining a row and a column of zeroes. Assume that A had no zero row, and that the smallest entry in A occurs the most times in the first row.

If $\tilde{A}$ has Kapranov rank two, then there is a rank-2 lift F of $\tilde{A}$ in which every 2×2 minor is non-singular and the i-th row can be written as a linear combination $\lambda_i u_i + \mu_i v_i$ of the first two rows u_1 and u_2, with $\deg(\lambda_i) \geq \deg(\mu_i) = 0$.

Proof. Starting with an arbitrary rank-2 lift, let F be obtained by adding to every row/column a linear combination of all other rows/columns with coefficients of sufficiently high degree but otherwise generic. This preserves the rank 2 of the lift and the degree of every entry, but makes every 2×2 minor of F non-singular.

In particular, this makes row i of F equal to a linear combination $\lambda_i u_1 + \mu_i u_2$ of the first two rows. If the degrees of λ_i and μ_i are different, then the minimum of them is zero in order to get a cancellation in the first column. But then $\deg(\mu_i) > \deg(\lambda_i) = 0$ is impossible, because it would correspond to a zero row in A. Hence, in this case $\deg(\lambda_i) > \deg(\mu_i) = 0$.

If the degrees are equal, then they are non-positive in order to get degree zero for the first entry in $\lambda_i u_1 + \mu_i u_2$. But they cannot be equal and negative, or otherwise entries of positive degree in u_2 would produce entries of negative degree in u_i. Hence, $\deg(\lambda_i) = \deg(\mu_i) = 0$ in this case. □

Corollary 34. *Let A and B be non-negative matrices. Assume that the two matrices*

$$\tilde{A} := \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} \text{ and } \tilde{B} := \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}$$

have Kapranov rank equal to 2. Then, the matrix

$$M := \begin{pmatrix} A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B \end{pmatrix}$$

has Kapranov rank equal to 2 as well.

Proof. Observe first that there is no loss of generality in assuming that A and B do not have zero rows. Hence the previous Lemma applies to them.

Let us number the rows of M from $-k$ to k' and its columns from $-l$ to l', where $k \times l$ and $k' \times l'$ are the dimensions of A and B respectively. In this way, A (respectively B) is the minor of negative (respectively, positive) indices. The row and column indexed zero consist of all zeroes. To further exhibit the symmetry between A and B the columns and rows in $\tilde{A}$ will be referred to "in reverse". That is to say, the first and second rows of it are the ones indexed 0 and -1 in M.

We now construct a lifting $M(t) = (a_{i,j}(t))$ of M. We assume that we are given rank-2 lifts of $\tilde{A}$ and $\tilde{B}$ in the conditions of the previous lemma. Furthermore, we assume that the lift of the entry $(0,0)$ is the same in both, which can be achieved by scaling the first row in one of them.

We use exactly those lifts of $\tilde{A}$ and $\tilde{B}$ for the upper-left and bottom-right corner minors of M.Our task is to complete that with an entry $a_{i,j}$ for every i,j with $ij < 0$, such that $\deg(a_{i,j}) = 0$ and the whole matrix still has rank 2. We claim that it suffices to choose the entry $a_{-1,1}$ of degree zero and sufficiently generic. That this choice fixes the rest of the matrix is easy. The entry $a_{1,-1}$ is fixed by the fact that the 3×3 minor

$$\begin{pmatrix} a_{-1,-1} & a_{-1,0} & a_{-1,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{1,-1} & a_{1,0} & a_{1,1} \end{pmatrix}$$

needs to have rank 2. All other entries $a_{-1,j}$ and $a_{1,j}$ are fixed by the fact that the entries $a_{-1,j}$, $a_{0,j}$ and $a_{1,j}$ (two of which come from either $\tilde{A}$ or $\tilde{B}$) must satisfy the same dependence as the three rows of the minor above. For each $i = -k, \dots, -2$ (respectively $i = 2, \dots, k'$), let λ_i and μ_i be the coefficients in the expression of the ith row of $\tilde{A}$ (respectively, of $\tilde{B}$) as $\lambda_i u_0 + \mu_i u_1$ (respectively, $\lambda_i u_0 + \mu_i u_{-1}$). Then, $a_{i,j} = \lambda_i + \mu_i a_{1,j}$ (respectively, $a_{i,j} = \lambda_i + \mu_i a_{-1,j}$).

We only need to show that if $a_{-1,1}$ and $a_{1,1}$ are of degree zero and sufficiently generic then all the new entries are of degree zero too. For this, observe that if $i \in \{-k', \dots, -2, 2, \dots, k\}$ then $a_{i,j}$ is of degree zero as long as the coefficient of degree zero in $a_{1,j}$ and $a_{-1,j}$ is different from the degree zero coefficients in all the quotients $-\lambda_i/\mu_i$ (here we are using the assumption that $\deg(\lambda_i) \geq \deg(\mu_i) \geq 0$). In terms of the choice of $a_{-1,1}(t)$ and $a_{1,-1}(t)$ this translates to the following determinants having non-zero coefficient in degree zero:

$$\begin{pmatrix} a_{-1,j} & a_{-1,-1} & a_{-1,0} \\ a_{0,j} & a_{0,-1} & a_{0,0} \\ -\lambda_i/\mu_i & a_{1,-1} & a_{1,0} \end{pmatrix}, \qquad \begin{pmatrix} a_{-1,0} & a_{-1,1} & -\lambda_i/\mu_i \\ a_{0,0} & a_{0,1} & a_{0,j} \\ a_{1,0} & a_{1,1} & a_{1,j} \end{pmatrix}$$

That $a_{-1,1}$ and $a_{1,-1}$ sufficiently generic imply non-singularity of these matrices follows from the fact that the following 2×2 minors come from the given lifts of $\tilde{A}$ and $\tilde{B}$, hence they are non-singular:

$$\begin{pmatrix} a_{-1,j} & a_{-1,0} \\ a_{0,j} & a_{0,0} \end{pmatrix}, \qquad \begin{pmatrix} a_{0,0} & a_{0,j} \\ a_{1,0} & a_{1,j} \end{pmatrix}.$$

□

Theorem 35. *Let M be a matrix of tropical rank equal to 2. Then its Kapranov rank is equal to 2 as well.*

Proof. The Kapranov rank of M is always at least the tropical rank, so we need only show that the Kapranov rank is less than or equal to 2. If the tropical convex hull of the rows of M is a path, then M has Barvinok rank 2 (Proposition 31) and thus Kapranov rank 2.

Otherwise, take x to be a node of degree at least three in this tropical convex hull, and apply the method of Lemma 32. Since x has degree at least three, it follows that there are at least three blocks A_i. We induct on the number of rows of M, taking three rows as the base case: If M has exactly three rows, then it must have no rows of all zeroes, and each block A_i must be a single row; in other words, every column has at most one positive entry. It is easy to construct an explicit lift of rank 2: in each column, lift the positive entry α as $-t^\alpha$ and the zero entries as -1 and $1+t^\alpha$ (if there are columns of zeroes, lift them as $(-1,-1,2)$, for example).

Next, suppose that M has $m \geq 4$ rows. The two blocks with the fewest number of combined rows have at least 2 and at most $m-2$ rows all together. Possibly after adding a row and column of zeroes, this provides a decomposition of our matrix as

$$M := \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B \end{pmatrix},$$

where both A and B have at least two rows. It then follows that the minors

$$\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}$$

have both less rows than the original matrix. By the inductive hypothesis they have Kapranov rank 2. Applying Corollary 34 completes the inductive step of the theorem. □

For Theorem 35 we again required the ability to pick generic coordinates, in the proof of Lemma 33. Thus, Theorem 35 holds over any infinite field, but may fail for instance over the finite field $\mathbb{F}_2$. This is illustrated by the following example. Observe that Proposition 22 and Theorem 28 fail here too, as well as the fact that Kapranov rank (over an arbitrary field) is invariant under insertion of a tropical combination of existing rows.

Example 36. *The matrix*

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

has Barvinok and tropical ranks equal to 2, but Kapranov rank 3 over $\mathbb{F}_2$.

7. Matrices constructed from matroids

One of the important properties of rank in usual linear algebra is that it produces a matroid. Unfortunately, the definitions of tropical rank, Kapranov rank, and Barvinok rank all fail to do this. Consider the configuration of four points $\{v_1,\ldots,v_4\}$ in the tropical plane given by the rows of the matrix

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix}.$$

Then by any of the three definitions above, the maximal independent sets are $\{1,2\}$, $\{1,3,4\}$, and $\{2,3,4\}$. These do not all have the same size, and so they cannot be the bases of a matroid. The central obstruction here is that the sets $\{1,2,3\}$ and $\{1,2,4\}$ are collinear, but the set $\{1,2,3,4\}$ is not. One possible alternate definition of independence is that a set of points is independent if they are generically placed (no tropical subdeterminant vanishes, which means that the tropical polytope has the "right" f-vector; see [7]), but this also fails, as in the above example the maximal independent sets are $\{1, 3, 4\}$ and $\{2, 4\}$. However, despite this failure, there is a strong connection between tropical linear algebra and matroids; we will use this connection to obtain matrices whose Kapranov rank exceeds their tropical rank.

The results in Sections 5 and 6 imply that any matrix whose tropical and Kapranov ranks must be at least of size 5×5. The smallest example we know of is 7×7 and it is based on the Fano matroid. To explain the example, and to show how to construct many others, we prove a theorem about tropical representations of matroids. The reader is referred to [10] for the appropriate matroid definitions.

Definition 37. *Let $\mathcal{M}$ be a matroid. The* cocircuit matrix *of $\mathcal{M}$, denoted $\mathcal{C}(\mathcal{M})$, has rows indexed by the elements of the ground set of $\mathcal{M}$ and columns indexed by the cocircuits of $\mathcal{M}$. It has a 0 in entry ij if the i-th element is in the j-th cocircuit and a 1 otherwise.*

In other words, $\mathcal{C}(\mathcal{M})$ is the 0/1 matrix whose columns have the cocircuits of $\mathcal{M}$ as supports (as before, the support of a column is its set of zeroes. Observe that this convention is opposite to the standard one in matroid theory). As an example, the matrix C_n of Section 2 is the tropical matrix of the uniform matroid of rank 2 with n elements. Similarly, the tropical matrix of the uniform matroid $U_{n,r}$ has size $n \times \binom{n}{r-1}$ and its columns are all the 0/1 vectors with exactly $r - 1$ ones. The following results show that its tropical and Kapranov ranks equal r. The tropical polytopes defined by these matrices are called "tropical hypersimplices" in [8], where they play a crucial role.

Proposition 38. *The tropical rank of the matrix $\mathcal{C}(\mathcal{M})$ is the rank of the matroid $\mathcal{M}$.*

Proof. This is a special case of Proposition 23. Indeed, the rank of a matroid is the maximum length of a chain of non-zero covectors, and the covectors (compositions of cocircuits) have supports equal to the unions of supports of cocircuits.

Observe that $\mathcal{C}(\mathcal{M})$ may have rows of ones, corresponding to loops in $\mathcal{M}$. But deleting them does not change neither the rank of $\mathcal{M}$ nor the tropical rank of $\mathcal{C}(\mathcal{M})$, the latter because every column has at least a zero and hence the origin lies in the tropical convex hull of the rows of $\mathcal{M}$. $\square$

As for the Kapranov rank, we can make a precise statement covering arbitrary infinite fields.

Proposition 39. *Let k be an infinite field. Then the Kapranov rank of $\mathcal{C}(\mathcal{M})$ (over k) is equal to the rank of $\mathcal{M}$ if and only if $\mathcal{M}$ is representable over k.*

Proof. Let the rank of $\mathcal{M}$ be r. We only need to show that there exists a lift of $\mathcal{C}(\mathcal{M})$ of rank r if and only if $\mathcal{M}$ is representable. Suppose that F is a lift of $\mathcal{C}(\mathcal{M})$ of rank r. For each row f_i of F let v_i be the vector of constant terms in f_i. We claim that the vectors $v_1, \dots, v_m$ are a representation of $\mathcal{M}$.

First, we claim that this vector configuration has rank at most r; pick r rows $f_1, \dots, f_r$ of F which span its row space. Then the coefficients in the linear combinations of these rows which produce the other rows must have leading exponent non-negative. Indeed, if the lowest coefficient is negative, then picking some cocircuit containing only that element among the r chosen ones yields a negative leading exponent for that entry. It then follows immediately that v_m is a linear combination of $v_1, \dots, v_r$, with coefficients equal to the constant terms of the coefficients of f_m as a linear combination of $f_1, \dots, f_r$.

Next, we claim that r of them form a basis in $\mathcal{M}$ if and only if they form a basis of the row space of F. Suppose $\{m_1, \dots, m_r\}$ is a basis of $\mathcal{M}$. Then, as in the proof of Proposition 38, we can find a square submatrix of $\mathcal{C}(\mathcal{M})$ using rows $1, \dots, r$ with 0's along the diagonal and 1's below it. However, this means that the lifted submatrix of constant terms is upper-triangular with nonzero entries along the diagonal, which immediately implies that $v_1, \dots, v_r$ are linearly independent as desired, and also proves that the represented matroid we constructed has rank exactly r.

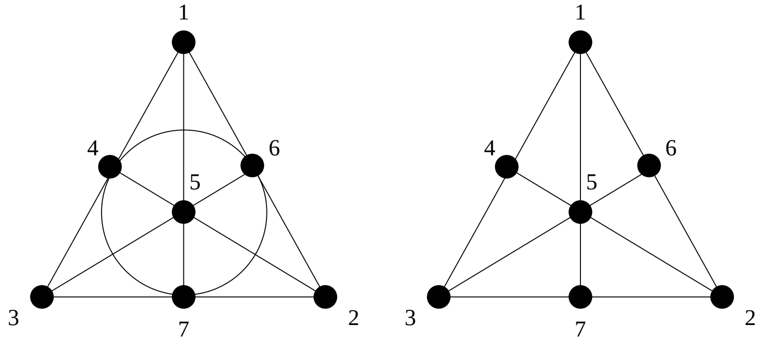

Figure 1. The Fano (left) and non-Fano (right) matroids.

If r points $m_1, \ldots, m_r$ are not a basis in $\mathcal{M}$, there exists a cocircuit containing none of them; this means that some column in $\mathcal{C}(\mathcal{M})$ has all 1's in rows $1, \ldots, r$. Therefore, in the lift, $f_1, \ldots, f_r$ all have no constant term in that column, and so $v_1, \ldots, v_r$ are all 0 in that coordinate. Since not all vectors v_i have an entry of 0 in that coordinate, $\{v_1, \ldots, v_r\}$ cannot be a basis.

This proves that if the rank of a matroid is equal to the Kapranov rank of its cocircuit matrix, the matroid is representable. For the other direction, assume the matroid is representable over k, and apply a linear transformation to assume that some basis $\{m_1, \ldots, m_r\}$ of $\mathcal{M}$ is represented by the standard basis $\{e_1, \ldots, e_r\}$. Let m_j be represented by the point v_j in the vector space V.

Cocircuits are complements of hyperplanes. For each hyperplane $\sum c_i x_i = 0$, lift the corresponding column of $\mathcal{C}(\mathcal{M})$ on the row corresponding to m_j to something with constant term $\sum c_i v_{ji}$, where v_{ji} is the i-th coordinate of the point v_j. Note that this gets lifted to 0 precisely when v_j is in the hyperplane, i.e. when m_j is not in the cocircuit. So if we fill things in with higher-order terms such that the t-coefficient of each entry in the lifted matrix is always nonzero, we will have obtained a bona fide lift of $\mathcal{C}(\mathcal{M})$.

The lifted matrix of constant terms clearly has rank r, since the first r rows are linearly independent and span the row space of the lift (each column being a linear functional on V and the first r rows spanning V.) So all we need to do is fill in higher-order terms (t and above) so that no t-coefficient is 0 and the rank condition is preserved. This is easy; simply pick $k_1, \ldots, k_r \in k$ so that $\sum k_i v_{ji}$ is never zero for any representing point v_j. This requires picking $(k_1, \ldots, k_r)$ to avoid a finite number of proper linear subspaces, which we can do since our field is infinite.

Now, simply let the t-coefficient of each entry of the j-th row be $\sum k_i v_{ji}$. This is also a linear function on V, and so our matrix still has rank r; we can write the j-th row y_j as $\sum_{i=1}^{r} v_{ji} y_i$, in particular. Therefore, we have constructed a lift of the tropical matrix $\mathcal{C}(\mathcal{M})$ which has rank r, completing the proof of Proposition 39. ☐

These two propositions have the following corollary, which gives us a whole slew of examples of matrices (over any field) whose Kapranov rank exceeds their tropical rank.

Corollary 40. *Let $\mathcal{M}$ be a matroid which is not representable over a field k. Then the Kapranov rank with respect to k of the tropical matrix $\mathcal{C}(\mathcal{M})$ exceeds its tropical rank.*

Consider, for example, the Fano and non-Fano matroids, depicted in Figure 1. They both have rank three and seven elements. The first is only representable over fields of characteristic two, the seond only over fields of characteristic different from two. In particular, they show that in every characteristic there are matrices with tropical rank equal to three and Kapranov rank larger than that. The cocircuit matrix of the Fano matroid is

$$\mathcal{C}(\mathcal{M}) = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This matrix is the smallest known example of a matrix whose Kapranov rank over $\mathbb{C}$ (four) is strictly larger than its tropical rank (three). Put differently, the seven rows of this matrix (in $\mathbb{TP}^6$) have as their tropical convex hull a two-dimensional cell complex which does not lie in any two-dimensional linear subspace of $\mathbb{TP}^6$, a feature decidedly absent from ordinary geometry. Note that Proposition 39 implies that the Kapranov rank of a matrix can be different for different ground fields, even when both are infinite; since the algebro-geometric Kapranov rank naturally associates with algebraically closed fields, this is a more significant discrepancy than that of Example 36, which used a finite field.

Combinatorially and geometrically, the cell complex in question is the Bergman complex of the Fano matroid (or rather the intersection of this pointed positive fan with $[0,1]^n$.) Indeed, this is true in general; Ardila and Klivans [2] showed that the Bergman complex of a matroid is naturally equal to the order complex of its lattice of flats, and using the fact that cocircuits are complements of hyperplanes, this is easily shown to be equal to the (dilated) tropical convex hull of the columns of $\mathcal{C}(\mathcal{M})$, leading to the following theorem.

Theorem 41. *The Bergman complex of the matroid $\mathcal{M}$ is equal to the tropical convex hull of the columns of the modified cocircuit matrix $\mathcal{C}'(\mathcal{M})$, where the 1's in $\mathcal{C}(\mathcal{M})$ are replaced by ∞'s.*

For the Fano matroid, this is the cone over the incidence graph of points and lines in the matroid. It consists of 15 vertices, 35 edges and 21 triangles.

Corollary 40 also naturally yields matrices with different Kapranov and tropical ranks over *every* field,if one starts with a non-representable matroid, such as the Vamos matroid (rank 4, 8 elements, 41 cocircuits) or the non-Pappus matroid (rank 3, 9 elements, 20 cocircuits).

One can also easily get examples in which the difference of the two ranks is arbitrarily large. Indeed, from given matrices A and B, we can construct the matrix

$$M := \begin{pmatrix} A & \infty \\ \infty' & B \end{pmatrix}$$

where ∞ and ∞' denote matrices of the appropriate dimensions and whose entries are sufficiently large. Then, the tropical and Kapranov ranks of M are the sums of those of (A, ∞) and (∞', B). By appropriate choice of these large values (picking those extra columns to be points in the tropical convex hull of the columns of A and B, and adding large constants to each column), we can ensure that these ranks are the same as those of A and B, which means that the difference between the Kapranov and tropical ranks of M is equal to the sum of this difference for A and for B.

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