ON THE ASSOCIATED GRAPHS TO A COMMUTATIVE RING

Z. BARATI, K. KHASHYARMANESH*, F. MOHAMMADI AND KH. NAFAR

Abstract. Let \( R \) be a commutative ring with non-zero identity. For an arbitrary multiplicatively closed subset \( S \) of \( R \), we associate a simple graph denoted by \( \Gamma_S(R) \) with all elements of \( R \) as vertices, and two distinct vertices \( x, y \in R \) are adjacent if and only if \( x + y \in S \). Two well-known graphs of this type are the total graph and the unit graph. In this paper, we study some basic properties of \( \Gamma_S(R) \). Moreover, we will improve and generalize some results for the total and the unit graphs.

Introduction

Throughout the paper \((R, +, \cdot)\) is a commutative ring with non-zero identity. We denote the set of zero-divisors and unit elements of \( R \) by \( Z(R) \) and \( U(R) \), respectively.

Finding the relationship between the algebraic structure of rings using properties of graphs associated to them has become an interesting topic in the last years. Indeed, it is worthwhile to relate algebraic properties of the rings to the combinatorical properties of the assigned graphs. One of the associated graphs to a ring \( R \) is the zero-divisor graph; it is a simple graph with vertex set \( Z(R) \setminus \{0\} \), and two vertices \( x \) and \( y \) are adjacent if and only if \( xy = 0 \), see [2, 4]. This graph was first introduced by Beck, in [8], where all the elements of \( R \) are considered as the vertices.

Anderson and Badawi, in [3], introduced the total graph of \( R \), as the simple graph with all elements of \( R \) as vertices, and two distinct vertices \( x \) and \( y \) are adjacent if and only if their sum is a zero-divisor. Recently, in [6], the authors considered the unit graph of \( R \), as the simple graph with all elements of \( R \) as vertices, and two distinct vertices \( x \) and \( y \) are adjacent if and only if \( x + y \in U(R) \).

A subset \( S \) of \( R \) which is closed under multiplication is called multiplicatively closed. In this paper, we introduce the graph \( \Gamma_S(R) \) associated to a ring \( R \) and a multiplicatively closed subset \( S \) of \( R \). The graph \( \Gamma_S(R) \) is a simple graph with all elements of \( R \) as vertices, and two distinct vertices \( x \) and \( y \) of \( R \) are adjacent if and only if \( x + y \in S \). Since the subsets \( Z(R) \) and \( U(R) \) of \( R \) are multiplicatively closed, \( \Gamma_S(R) \) is a natural generalization of the last two graphs. Hence the total graph and the unit graph of \( R \) are some well-known graphs of this type.

In the first section, we observe the relationship of the associated graphs \( \Gamma_S(R) \) with the total graph, unit graph, and some Cayley graphs. Then we study the basic properties of the associated graphs, like the degree of the vertices and connectivity.

1991 Mathematics Subject Classification. 05C75, 13A10, 13A15.

Key words and phrases. Commutative rings, Unit graph, Total graph, Diameter, Girth.

*Corresponding author.
Our results improve and generalize some corresponding results for the total graph and the unit graph which were studied in [3] and [6]. Also, it is worthwhile to study the relationship between the associated graphs and the multiplicatively closed subsets $S$ and $S^c$ (see Theorem 1.9).

In the next section, we consider the case that $S$ is a saturated multiplicatively closed subset of $R$. A multiplicatively closed subset $S$ of $R$ is called saturated if $xy \in S$ implies that $x \in S$ and $y \in S$. For a non-empty saturated multiplicatively closed subset $S$ and an arbitrary element $s \in S$, we have $s \cdot 1 \in S$, which implies that $1 \in S$. This means that $U(R) \subseteq S$. By [9, Page 2, Theorem 2] a subset $S$ of $R$ is saturated if and only if $R \setminus S$ is a union of some prime ideals. Hence $R \setminus S = \bigcup_{i \in A} p_i$ for some prime ideals $p_i$ with $i \in A$. Set $I := \cap_{i \in A} p_i$ and $\tilde{S} := \{s + I : s \in S\}$. We observe that the newly constructed subset $\tilde{S}$ is a saturated multiplicatively closed subset of $R/I$. Then we study the relationship between the combinatorial properties of the graphs $\Gamma_S(R)$ and $\Gamma_{\tilde{S}}(R/I)$.

Next, we check the possible integers for the girth and the diameter of the graph $\Gamma_S(R)$ and we observe that these are exactly the integers which can be realized as the girth and the diameter of the unit graphs (see Theorems 2.15, 2.23).

Also, in this paper, we will focus on the idealization of an $R$-module $M$ over $R$, denoted by $R(+)^M$, which is a commutative ring formed from $R \times M$ with addition and multiplication as $(r,m) + (s,n) = (r + s, m + n)$ and $(r,m)(s,n) = (rs, rn + sm)$, respectively. We will describe the associated graph to the ring $R(+)^M$ and an appropriate saturated multiplicatively closed subset of $R(+)^M$ in terms of the associated graphs to some edge-induced subgraphs of $G$.

Let $G$ be a graph. For two arbitrary vertices $a$ and $b$ of $G$, a path of length $r$ between $a$ and $b$ is an ordered list of distinct vertices $a = x_0, x_1, \ldots, x_n = b$ such that $x_{i-1}x_i$ are edges for all $i = 1, \ldots, n$. We denote a path between $a$ and $b$ by $a = x_0 \to x_1 \to \cdots \to x_n = b$. Note that the considered graphs are simple and the above notion for a path is just for simplifying the cycles. A cycle is a path $x_0, x_1, \ldots, x_n$ with the extra edge $x_nx_0$. For two vertices $a$ and $b$ of $G$, the length of a shortest path from $a$ to $b$ is denoted by $d(a,b)$. Note that if there is no path of finite length between $a$ and $b$, then $d(a,b) = \infty$. The diameter of $G$ is defined as $\text{diam}(G) = \sup\{d(a,b) : a \text{ and } b \text{ are vertices of } G\}$ and the girth of $G$, denoted by $\text{gr}(G)$, is the smaller integer $n$ such that there exists a cycle $x_0, x_1, \ldots, x_n$ in $G$. If there is no cycle of finite length in $G$, then $\text{gr}(G) = \infty$. A graph $G$ is connected if there exists a path between every two vertices $a$ and $b$ of $G$. A bipartite graph is one whose vertices are partitioned into two disjoint parts such that the vertices of each edge belong to different partitions. A complete graph on the $n$ vertices, denoted by $K_n$, is a graph such that each pair of distinct vertices are adjacent.

1. Basic properties of the associated graphs

In this section, $S$ is an arbitrary multiplicatively closed subset of $R$. Our first example shows that, in general, $\Gamma_S(R)$ is not isomorphic to the total or unit graph.
Example 1.1. Let $R = \mathbb{Z}_6$ be the ring of integers modulo 6. Then $Z(R) = \{0, 2, 3, 4\}$ and $U(R) = \{1, 5\}$. The graphs $\Gamma_S(R)$ for $S = Z(R)$, $S = \{1, 3, 5\}$ and $S = U(R)$ are shown in the following figures.

Let $H$ be a finite group with identity element $e$ and let $T$ be a subset of $H$ such that $e \notin T$ and $T^{-1} = \{x^{-1} : x \in T\} \subseteq T$. Then the Cayley graph associated to $H$ and $T$, denoted by $\text{Cay}(H, T)$, is a simple graph with all elements of $H$ as vertices, and two distinct vertices $x$ and $y$ of $H$ are adjacent if and only if $xy^{-1} \in T$. In the following example, we show that, for a positive integer $n$, the Cayley graph $\text{Cay}(\mathbb{Z}_{2n}, \{1, -1\})$ is isomorphic to $\Gamma_{\{1, -1\}}(\mathbb{Z}_{2n})$.

Example 1.2. Let $R = \mathbb{Z}_{2n}$ and $S = \{1, -1\} \subseteq R$. Then $S$ is a multiplicatively closed subset of $R$. Consider $(\mathbb{Z}_{2n}, +)$ as a group, so we can define $\text{Cay}(\mathbb{Z}_{2n}, \{1, -1\})$. We will observe that two graphs $\Gamma_{\{1, -1\}}(\mathbb{Z}_{2n})$ and $\text{Cay}(\mathbb{Z}_{2n}, \{1, -1\})$ are isomorphic. To do this, consider the map

$$\varphi : \Gamma_{\{1, -1\}}(\mathbb{Z}_{2n}) \rightarrow \text{Cay}(\mathbb{Z}_{2n}, \{1, -1\})$$

given by $\varphi(x) = x$, if $x$ is even and $\varphi(x) = -x$, otherwise. Clearly $\varphi$ is a bijection. Now, we show that $\varphi$ is a homomorphism. To achieve this, suppose that $\{x, y\}$ is an edge in $\Gamma_{\{1, -1\}}(\mathbb{Z}_{2n})$. So both $x$ and $y$ are neither even nor odd. Hence without loss of generality, we may assume that $x$ is even and $y$ is odd and that the sum of $x$ and $y$ are 1 or $-1$. Therefore $x + (-y)$ is equal to 1 or $-1$. This means that $\varphi(x)$ and $\varphi(y)$ are adjacent in $\text{Cay}(\mathbb{Z}_{2n}, \{1, -1\})$. Conversely, if $\{x, y\}$ is an edge in $\text{Cay}(\mathbb{Z}_{2n}, \{1, -1\})$, then $x - y$ is equal to 1 or $-1$. Hence without loss of generality, we may assume that $x$ is odd and $y$ is even. Since $\varphi(-x) = x$ and $\varphi(y) = y$, we have that $-x - y$ is equal to 1 or $-1$, which implies that the corresponding vertices $x$ and $y$ are adjacent in $\Gamma_{\{1, -1\}}(\mathbb{Z}_{2n})$.

Recall that for a graph $G$ the neighbor set of a vertex $x$ of $G$ is the set $N_G(x) = \{y : y$ is adjacent to $x\}$. The number of vertices adjacent to $x$, denoted by $\deg(x)$, is called the degree of $x$.

Lemma 1.3. Suppose that $S$ is an arbitrary multiplicatively closed subset of $R$. Then, in the graph $\Gamma_S(R)$,

(a) for each $x \in R$ with $x + x \notin S$, we have $\deg(x) = |S|$, and

(b) for each $x \in R$ with $x + x \in S$, we have $\deg(x) = |S| - 1$. In particular, if $2 \in S$, then $\deg(x) = |S| - 1$ for all $x \in S$.  


**Proof.** For any vertex \( x \) of \( \Gamma_S(R) \), we have that \( N_{\Gamma_S(R)}(x) = \{ s - x : s \in S \} \). Now, if \( x + x \not\in S \), then \( \deg(x) = |N_{\Gamma_S(R)}(x)| = |S| \). Since \( \Gamma_S(R) \) is a simple graph, in the case \( x + x \in S \), one can conclude that \( \deg(x) = |S| - 1 \). \( \square \)

**Theorem 1.4.** Suppose that \( S \) is an ideal of \( R \) with \( |S| = \alpha \). Set \( A := \{ x + S : x \in R \setminus S \text{ and } 2x \in S \} \) and \( B := \{ x + S : x \in R \setminus S \text{ and } 2x \not\in S \} \). Then \( \Gamma_S(R) \) is the disjoint union of \( |A| + 1 \) times \( K_\alpha \) and \( |B|/2 \) times \( K_{\alpha, \alpha} \).

**Proof.** For each two elements \( x, y \) in \( S \), we have that \( x + y \in S \) and \( 2x \in S \). Now Lemma 1.3 (b) shows that the induced subgraph of \( \Gamma_S(R) \) on the vertices \( S \) is the complete graph \( K_\alpha \). Moreover, since \( S \) is an ideal, for each two vertices \( x \in S \) and \( y \in R \setminus S \), we have \( x + y \not\in S \). This means that the induced subgraph of \( \Gamma_S(R) \) on the vertices \( S \), \( K_\alpha \) is disjoint from the other vertices. In other words, \( K_\alpha \) is a connected component of \( \Gamma_S(R) \).

Assume that \( x + x \in S \). Since \( S \) is an ideal, \( (x + s_1) + (x + s_2) = 2x + (s_1 + s_2) \in S \) for all \( s_1, s_2 \in S \). This implies that the coset \( x + S \) is a complete subgraph of \( \Gamma_S(R) \). If two vertices \( x + s_1 \) and \( y + s_2 \) from distinct cosets are adjacent in \( \Gamma_S(R) \), then \( x + y = (x + s_1) + (y + s_2) - (s_1 + s_2) \in S \), and so \( y + S = -x + S \). On other hand \( 2x \in S \), and so \( x + s_1 = -x + s_2 \) for some \( s_2, s_3 \in S \). Therefore \( y + S = x + S \). This means that corresponding to each element \( x \) of \( A \), we have a complete graph \( K_\alpha \) which is disjoint from the other vertices.

Now, suppose that \( 2x \not\in S \). Then, for all \( s_1, s_2 \in S \), it is easy to see that \( (x + s_1) + (x + s_2) = 2x + (s_1 + s_2) \not\in S \). Also, each element of the coset \( x + S \) is adjacent to each element of the coset \( -x + S \). Therefore, corresponding to each \( x \in B \), we have a complete bipartite graph \( K_{\alpha, \alpha} \) on the vertex set \( x + S \cup -x + S \). On the other hand, if the elements \( x + s_1 \) and \( y + s_1 \) from distinct cosets are adjacent in \( \Gamma_S(R) \), then \( x + y = (x + s_1) + (y + s_2) - (s_1 + s_2) \in S \), and so \( y + S = -x + S \). Therefore each complete bipartite graph \( K_{\alpha, \alpha} \) on the vertex set \( x + S \cup -x + S \) is disjoint from the other vertices.

These facts imply that \( \Gamma_S(R) \) is the disjoint union of \( |A| + 1 \) times \( K_\alpha \) and \( |B|/2 \) times \( K_{\alpha, \alpha} \), as desired. \( \square \)

**Example 1.5.** Let \( R = \mathbb{Z}_{12} \) and \( S = \{0, 6\} \subseteq R \). Then, by using the notation in Theorem 1.4, \( A = \{3 + S\} \) and \( B = \{1 + S, 4 + S, 8 + S, 11 + S\} \). Note that \( 3 + S = 9 + S \). As we show in the following figure, the graph \( \Gamma_S(R) \) is the disjoint union of two \( K_2 \) components and two \( K_{2, 2} \) components.

![Graph](figure.png)

As a consequence of Theorem 1.4, we have the following corollary which is an improved form of [3, Theorem 2.2].
Corollary 1.6. Suppose that $S$ is an ideal of $R$ with $|S| = \alpha$ and $|R/S| = \beta$.

(a) If $2 \in S$, then $\Gamma_S(R)$ is the union of $\beta$ disjoint $K_\alpha$'s.
(b) If $2x \notin S$ for each $x \in R \setminus S$, then $\Gamma_S(R)$ is the union of $K_\alpha$ with $(\beta - 1)/2$ disjoint $K_{\alpha,a}$'s.

Proof. If $2 \in S$, then $R = \bigcup_{x \in A}(x + S) \cup S$ and $|A| + 1 = \beta$. If $2 \notin S$, then $R = \bigcup_{x \in B}(x + S) \cup S$ and $|B| + 1 = \beta$. Now the results follow from Theorem 1.4. □

Now, we are going to find a necessary and sufficient condition for the connectedness of the graph $\Gamma_S(R)$ in the case that $S = -S$. Since $Z(R)$ and $U(R)$ fulfill this condition, the following theorem can be considered as an improved form of [3, Theorem 3.3] and [10, Proposition 3.2].

Theorem 1.7. Let $S$ be a multiplicatively closed subset of $R$ such that $S = -S$. Then $\Gamma_S(R)$ is connected if and only if $(R, +)$ is generated by $S$.

Proof. Let $\Gamma_S(R)$ be a connected graph. For $a \in R$ and $s \in S$, there exists a path

$$s \to x_1 \to x_2 \to \cdots \to x_n \to a$$

from $s$ to $a$ in $\Gamma_S(R)$. Hence the elements $s + x_1, x_1 + x_2, \ldots, x_n + a$ are in $S$. If $n$ is an odd number, then

$$a = s - (s + x_1) + (x_1 + x_2) - \cdots + (x_n + a),$$

and if $n$ is an even number, then

$$a = -s + (s + x_1) - (x_1 + x_2) + \cdots + (x_n + a).$$

Therefore $a$ can be written as the sum of the elements of $S$, as desired.

Conversely, suppose that $R$ is generated by $S$. It is enough to find a path from 0 to each vertex $x$ of $\Gamma_S(R)$. For a non-zero element $x$ in $R$, there exist some elements $s_1, s_2, \ldots, s_n$ of $S$ with $x = s_1 + s_2 + \cdots + s_n$. Then

$$0 \to (-1)^{n+1}s_1 \to (-1)^{n+2}(s_1 + s_2) \to \cdots \to (-1)^{n+n}(s_1 + s_2 + \cdots + s_n) = x$$

is an expected path from 0 to $x$. □

Note that $S$ in Theorem 1.7 is not necessarily an ideal of $R$. For instance, if $R = \mathbb{Z}$ and $S = \{1, -1\}$, then $\Gamma_S(R)$ is connected.

Corollary 1.8. For a proper ideal $S$ of $R$, the graph $\Gamma_S(R)$ is disconnected.

In the case that both subsets $S$ and $S^c = R \setminus S$ of $R$ are multiplicatively closed, we have the following description for the graphs associated to them.

Theorem 1.9. Suppose that $S$ and $S^c = R \setminus S$ are two multiplicatively closed subsets of $R$. Then the complement of $\Gamma_S(R)$ is isomorphic to $\Gamma_{S^c}(R)$.

Proof. First note that the vertices of $\Gamma_S(R)$ and $\Gamma_{S^c}(R)$ are all the elements of $R$. Two vertices $x$ and $y$ are not adjacent in $\Gamma_S(R)$ if and only if $x + y \notin S$ which is equivalent to the vertices $x$ and $y$ are adjacent in $\Gamma_{S^c}(R)$. □
As a consequence of Theorem 1.9, we observe the relationship between the unit graph and the total graph of a finite ring.

**Corollary 1.10.** The complement of the unit graph of a finite ring $R$ is isomorphic to its total graph.

**Proof.** Since $R$ is finite, it is the disjoint union of $Z(R)$ and $U(R)$. The result now follows from Theorem 1.9. □

Suppose that $M$ is an $R$-module. As mentioned in the introduction, the idealization $R(+)M$ of $M$ over $R$ is a commutative ring. If $S$ is a multiplicatively closed subset of $R$, then it is easy to see that $\hat{S} := S(+)M$ is a multiplicatively closed subset of $R(+)M$. The following theorem compares the diameter of the graph $\Gamma_{\hat{S}}(R(+)M)$ with the diameter of $\Gamma_S(R)$.

**Theorem 1.11.** For an $R$-module $M$, $\Gamma_{\hat{S}}(R(+)M)$ is connected if and only if $\Gamma_S(R)$ is connected. More precisely,

$$\text{diam}(\Gamma_{\hat{S}}(R(+)M)) = \text{diam}(\Gamma_S(R)).$$

**Proof.** Suppose that $\Gamma_{\hat{S}}(R(+)M)$ is connected. For $x, y \in R$, consider the corresponding elements $(x, 0)$ and $(y, 0)$ in $R(+)M$. By our assumption, there exists a path

$$(x, 0) \to (s_1, m_1) \to \cdots \to (s_n, t_n) \to (y, 0)$$

in $\Gamma_{\hat{S}}(R(+)M)$, and so the corresponding path $x \to s_1 \to \cdots \to s_n \to y$ is the desired path in $\Gamma_S(R)$.

Now, suppose that $\Gamma_S(R)$ is connected and $(x, m_1)$ and $(y, m_2)$ are arbitrary elements in $R(+)M$. First assume that $x = y$. Then, for arbitrary element $s \in S$

$$(x, m_1) \to (x + s, 0) \to (y, m_2)$$

is a path in $\Gamma_{\hat{S}}(R(+)M)$. If $x \neq y$, then there exist elements $s_1, s_2, \ldots, s_n$ in $S$ such that $x \to s_1 \to \cdots \to s_n \to y$ is a path in $\Gamma_S(R)$. Thus

$$(x, m_1) \to (s_1, 0) \to \cdots \to (s_n, 0) \to (y, m_2)$$

is the desired path in $\Gamma_{\hat{S}}(R(+)M)$, which completes the proof. □

As explained in [3], in general, $Z(R)(+)M \subseteq Z(R(+)M)$. Hence Theorem 1.11 is an improved from of [3, Theorem 3.16], where the authors considered the case that the equality $Z(R)(+)M = Z(R(+)M)$ holds.

2. **Graphs associated to saturated multiplicatively closed subsets**

The set of all unit elements $U(R)$ of $R$ is a saturated multiplicatively closed subset of $R$. On the other hand, if $S$ is a saturated multiplicatively closed subset of $R$, then, for an arbitrary element $s$ in $S$, $1 \cdot s \in S$. Hence $1 \in S$. Thus, for any $u \in U(R)$, $uu^{-1} = 1 \in S$, which implies that $u \in S$, and so $U(R) \subseteq S$. This means that $U(R)$ is the smallest saturated multiplicatively closed subset of $R$. In this section, $S$ is a saturated multiplicatively closed subset of $R$, and so our results about the graph $\Gamma_S(R)$ are natural generalizations of the corresponding results for the unit graph. The following results provide a characterization for the completeness of the graphs $\Gamma_S(R)$.
**Proposition 2.1.** The graph $\Gamma_S(R)$ is complete if and only if $S = R$ or $\text{char } R = 2$ and $S = R \setminus \{0\}$.

**Proof.** Let $\Gamma_S(R)$ be a complete graph and $S \neq R$. Then, for every non-zero element $x \in R$, $x$ is adjacent to 0, and so $x \in S$. Hence $S = R \setminus \{0\}$. Since $0 \notin S$ and $\Gamma_S(R)$ is complete, $1 = -1$, and so char $R = 2$.

Conversely, suppose that $S = R \setminus \{0\}$ and char $R = 2$. So, for every distinct elements $x$ and $y$, we have that $x + y \neq 0$, and hence $x + y \in S$, which implies that $\Gamma_S(R)$ is a complete graph. $\square$

Note that if $0 \in S$, then $S = R$, and so, by Proposition 2.1, the graph $\Gamma_S(R)$ is complete. Hence in the rest of this section, we will assume that $0 \notin S$. Now, in view of [9, Page 2, Theorem 2], $R \setminus S = \bigcup_{i \in A} p_i$ for some prime ideals $p_i$ of $R$. Put $I := \bigcap_{i \in A} p_i$ and $\tilde{S} := \{s + I : s \in S\}$.

**Lemma 2.2.** By using the above notation, $\tilde{S}$ is a saturated multiplicatively closed subset of $R/I$.

**Proof.** Clearly $\tilde{S}$ is a multiplicatively closed subset of $R/I$. Let $a + I$ and $b + I$ be two elements of $R/I$ with $(a + I)(b + I) \in \tilde{S}$. Assume to the contrary that $a \notin S$. Then $ab \notin S$, and so for some $j \in A$, $ab \in p_j$. Moreover, since $(a + I)(b + I) \in \tilde{S}$, for some $s \in S$, we have that $ab - s \in p_j$. This implies that $s \in p_j$, which is impossible. Thus $\tilde{S}$ is a saturated multiplicatively closed subset of $R/I$. $\square$

As an immediate consequence of Lemma 1.3, we have the following lemma.

**Lemma 2.3.** For an arbitrary saturated multiplicatively closed subset $S$ of $R$, in the graph $\Gamma_S(R)$, the following statements hold.

(a) If $x \in R \setminus S$, then $\text{deg}(x) = |S|$.

(b) If $0 \notin S$, then $\text{deg}(x) = |S|$ for all $x \in R$.

**Lemma 2.4.** Let $x$ and $y$ be two elements of $R$. Then the following statements are equivalent:

(a) $x$ is adjacent to $y$ in $\Gamma_S(R)$;

(b) $x + I$ is adjacent to $y + I$ in $\Gamma_S(R/I)$;

(c) each element of $x + I$ is adjacent to each element of $y + I$ in $\Gamma_S(R)$; and,

(d) there exist $x + i$ in $x + I$ and $y + i'$ in $y + I$ which are adjacent in $\Gamma_S(R)$.

**Proof.** The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are clear.

(b) $\Rightarrow$ (c) Assume to the contrary that the elements $x + i$ in $x + I$ and $y + i'$ in $y + I$ are not adjacent in $\Gamma_S(R)$. Thus $x + y + i + i' \notin S$, and so $x + y + i + i' \in p_j$, and hence $x + y \in p_j$, for some $j \in A$. Since $x + I$ and $y + I$ are adjacent in $\Gamma_S(R/I)$, for some $s \in S$, we have that $x + y + s \in I$. Therefore $x + y + s \in p_j$, and so $s \in p_j$, which is the required contradiction.

(d) $\Rightarrow$ (a) Since $x + i$ and $y + i'$ are adjacent in $\Gamma_S(R)$, we have $x + y + i + i' \in S$. Now, if $x + y \notin S$, then $x + y \notin p_j$, for some $j \in A$. Hence $x + y + i + i' \in p_j$, which is impossible since $S \cap p_j = \emptyset$. Therefore $x + y \in S$, which implies that $x$ and $y$ are adjacent in $\Gamma_S(R)$. $\square$
As an easy consequence of Lemma 2.4, we have the following corollary.

**Corollary 2.5.** The graph $\Gamma_S(R)$ is connected if and only if $\tilde{\Gamma}_S(R/I)$ is connected.

Recall that a **clique** of a graph $G$ is a complete subgraph of $G$. Also, a co-clique in a graph $G$ is a set of pairwise non-adjacent vertices of $G$. In the following corollary, we study the existence of cliques (and co-cliques) in the graph $\Gamma_S(R)$.

**Corollary 2.6.** Let $R$ be a commutative ring. Then the following statements hold.

(a) If $2x \notin S$ for some $x \in R$, then $x + I$ is a co-clique in $\Gamma_S(R)$.
(b) If $2x \in S$ for some $x \in R$, then $x + I$ is a clique in $\Gamma_S(R)$.

**Proof.** (a) Suppose that $x + i$ and $x + i'$ are two arbitrary elements of $R$. If $x + i$ and $x + i'$ are adjacent in $\Gamma_S(R)$, then Lemma 2.4 (d) implies that $x + x \in S$, which is impossible. So $x + I$ is a co-clique in $\Gamma_S(R)$. Part (b) is an immediate consequence of Lemma 2.4 (c). \qed

In the following theorem, we would like to study the relationship between the diameter and the girth of the graphs $\Gamma_S(R)$ and $\tilde{\Gamma}_S(R/I)$.

**Theorem 2.7.** The following statements hold.

(a) $\text{gr}(\Gamma_S(R)) \leq \text{gr}(\tilde{\Gamma}_S(R/I))$;
(b) $\text{diam}(\Gamma_S(R/I)) \leq \text{diam}(\tilde{\Gamma}_S(R))$;
(c) If $\tilde{\Gamma}_S(R/I)$ is a complete graph, then $\text{diam}(\Gamma_S(R)) \leq 2$; and,
(d) If $\tilde{\Gamma}_S(R/I)$ is not a complete graph, then $\text{diam}(\Gamma_S(R)) = \text{diam}(\tilde{\Gamma}_S(R/I))$.

**Proof.** (a) If $\tilde{\Gamma}_S(R/I)$ has no cycle, then there is nothing to prove. So suppose that

$$x_0 + I \to x_1 + I \to \cdots \to x_t + I \to x_0 + I$$

is a cycle in $\tilde{\Gamma}_S(R/I)$. Then the elements $x_0 + x_1, x_1 + x_2, \ldots, x_t + x_0$ belong to $S$, and so $x_0 \to x_1 \to \cdots \to x_t \to x_0$ is a cycle in $\Gamma_S(R)$, which implies that $\text{gr}(\tilde{\Gamma}_S(R/I)) \leq \text{gr}(\Gamma_S(R))$.

(b) Set $t := \text{diam}(\Gamma_S(R/I))$, and let $a + I$ and $b + I$ be two vertices of $\Gamma_S(R/I)$ with $d(a + I, b + I) = t$. Assume that

$$a + I \to x_1 + I \to \cdots \to x_{t-1} + I \to b + I$$

is a corresponding path of length $t$ between $a + I$ and $b + I$ in $\Gamma_S(R/I)$. Since $a + x_1, x_1 + x_2, \ldots, x_{t-2} + x_{t-1}, x_{t-1} + b$ are elements in $S$, in view of Lemma 2.4, $a \to x_1 \to \cdots \to x_{t-1} \to b$ is a path of length $t$ in $\Gamma_S(R)$, as desired.

(c) We claim that, for every two nonadjacent vertices $x$ and $y$ of $\Gamma_S(R)$, $d(x, y) = 2$. Now, if $x + I \neq y + I$, then $x$ and $y$ are adjacent in $\Gamma_S(R)$. So we may assume that $x + I = y + I$. Since $\tilde{\Gamma}_S(R/I)$ has no isolated vertex, there exists a vertex $z + I$ adjacent to $x + I$ in $\tilde{\Gamma}_S(R/I)$. Thus $x \to z \to y$ is a path between $x$ and $y$ which implies that $d(x, y) = 2$. Therefore $\text{diam}(\Gamma_S(R)) \leq 2$.

(d) In the view of Corollary 2.5, we may assume that the $\Gamma_S(R)$ and $\tilde{\Gamma}_S(R/I)$ are connected. Put $m := \text{diam}(\Gamma_S(R))$ and $t := \text{diam}(\tilde{\Gamma}_S(R/I))$. Since $\tilde{\Gamma}_S(R/I)$ is not complete, $t > 1$. Also, by (c), $t \leq m$. Now, assume to the contrary that $t < m$. Let $x$ and $y$ be two vertices in $\Gamma_S(R)$ such that $d_{\Gamma_S(R)}(x, y) = m$. One can assume $x$ and $y$ lie in distinct cosets of $I$. Assume that
\[ x + I \to y_1 + I \to \cdots \to y_{n-1} + I \to y + I \]

is a path between \( x + I \) and \( y + I \) such that \( d_{\Gamma_S(R/I)}(x + I, y + I) = n \). So \( n \leq t \).

Clearly

\[ x \to y_1 \to \cdots \to y_{n-1} \to y \]

is a path of length \( n \) between \( x \) and \( y \) in the graph \( \Gamma_S(R) \), which is impossible since \( n \leq t < m \).

The following example shows that we may have strict inequality in part (a) of Theorem 2.7. In our example, \( \Gamma_S(R/I) \) is a complete graph, but \( \Gamma_S(R) \) is not a complete graph by Theorem 2.7 (c).

**Example 2.8.** Let \( R = \mathbb{Z}_2[x] \) be the polynomial ring in an indeterminate \( x \) with coefficients in \( \mathbb{Z}_2 \), and set \( S := R \setminus \langle x \rangle \). Hence \( I = \langle x \rangle \), and so \( |R/I| = 2 \). Set \( V_1 := \{a_1 x + \cdots + a_n x^n : n \in \mathbb{N} \text{ and } a_1, \ldots, a_n \in \mathbb{Z}_2 \} \) and \( V_2 := \{1 + b_1 x + \cdots + b_m x^m : m \in \mathbb{N} \text{ and } b_1, \ldots, b_m \in \mathbb{Z}_2 \} \). Clearly the sum of two elements of \( V_1 \) (or \( V_2 \)) is in \( \langle x \rangle \). This implies that the vertices in the same part are not adjacent. Moreover, for each two vertices \( f = a_1 x + \cdots + a_n x^n \in V_1 \) and \( g = 1 + b_1 x + \cdots + b_m x^m \in V_2 \), we have that \( f + g \in S \). This means that \( \Gamma_S(R) \) is a complete bipartite graph, and so \( \text{diam}(\Gamma_S(R)) = 2 \). On the other hand, \( \Gamma_S(R/I) \) is a \( K_2 \), and so \( \text{diam}(\Gamma_S(R/I)) = 1 \).

Hence the upper bound given in Theorem 2.7 (c) is sharp.

Moreover, one can see that \( \text{gr}(\Gamma_S(R/I)) = \infty \) and \( \text{gr}(\Gamma_S(R)) = 4 \). These facts show that the converse of Theorem 2.7 (a) does not hold. Hence, in general, the equality in part (a) of Theorem 2.7 does not hold.

Let \( G \) be a graph with edge set \( E(G) \). Also, suppose that there exists a family of edge-disjoint subgraphs \( \{G_i\}_{i \in I} \) of \( G \). Then we put \( G = \bigoplus_{i \in I} G_i \). Furthermore, in the case that \( G_i \cong H \) for every \( i \in I \), we set \( G = \bigoplus_{i \in I} H \).

Now, let \( R/I = \bigcup_{i \in A} (x_i + I) \) such that for any two indices \( i \) and \( j \), \( x_i + I \neq x_j + I \). Then \( \{x_i : i \in A\} \) is called a system of representation of \( R/I \). In the following proposition, we provide a characterization of \( \Gamma_S(R) \) in terms of \( \Gamma_S(R/I) \).

**Proposition 2.9.** The following statements hold.

(a) If \( 2 \notin S \), then \( \Gamma_S(R) = \bigoplus_{I^2} \Gamma_S(R/I) \).

(b) If \( 2 \in S \), then \( \Gamma_S(R) = (\bigoplus_{I^2} \Gamma_S(R/I)) \oplus (\bigoplus_{|C|} K_{|I|}) \), where \( C = \Delta \cap S \) for a system of representation \( \Delta \) of \( R/I \).

**Proof.** Let \( \Delta := \{x_i : i \in A\} \) be a system of representative of \( R/I \). For \( i, j \in A \), if \( x_i + I \) and \( x_j + I \) are adjacent in \( \Gamma_S(R/I) \), in view of Lemma 2.4, each element of \( x_i + I \) is adjacent to each element of \( x_j + I \) in \( \Gamma_S(R) \). Indeed, each edge of \( \Gamma_S(R/I) \) corresponds to exactly \( |I|^2 \) edges in \( \Gamma_S(R) \). Now, if \( 2 \notin S \), then \( 2x \notin S \) for each \( x \in R \), and so the coset \( x + I \) forms a co-clique in \( \Gamma_S(R) \). Hence \( \Gamma_S(R) = \bigoplus_{I^2} \Gamma_S(R/I) \), as desired. Also, whenever, \( 2 \in S \), then, for each element \( x \in S \), \( 2x \in S \). This implies that, for \( x \in S \), the vertex \( x + I \) in \( \Gamma_S(R/I) \) corresponds to a clique in \( \Gamma_S(R) \). Moreover, for each \( x \in S \), the equality \( x + I = y + I \) implies that \( y \in S \). Therefore \( \Gamma_S(R) = (\bigoplus_{I^2} \Gamma_S(R/I)) \oplus (\bigoplus_{|C|} K_{|I|}) \), where \( C = \Delta \cap S \). \( \square \)
In the following remark, we show that Proposition 2.9 is an improved form of [6, Proposition 2.8].

**Remark 2.10.** Let $R$ be a finite ring and $S := U(R)$. By [7, Proposition 1.10], $Z(R)$ is the union of the minimal associated prime ideals of the zero ideal of $R$. Since $R$ is Artinian, every prime ideal of $R$ is maximal, and so $Z(R) = R \setminus S = \bigcup_{p \in \text{Max}(R)} p$. Hence $I = J(R)$. This shows that Proposition 2.8 in [6] is a consequence of Proposition 2.9.

Let $S$ be a saturated multiplicatively closed subset of $R$ and $M$ be an $R$-module. One can easily check that $\hat{S} = S(+)M$ is a saturated multiplicatively closed subset of $R(+)M$. For every $r \in R$, put $A_r := \{(r, m) : m \in M\}$. Hence $R(+)M$ is the disjoint union of the $A_r$’s, i.e. $R(+)M = \bigcup_{r \in R} A_r$.

**Lemma 2.11.** For arbitrary elements $r, r' \in R$ and an $R$-module $M$, the following statements are equivalent:

(a) the vertex $r$ is adjacent to $r'$ in $\Gamma_S(R)$,

(b) every elements of $A_r$ is adjacent to every elements of $A_{r'}$ in $\Gamma_S(R(+)M)$, and

(c) an element of $A_r$ is adjacent to an element of $A_{r'}$ in $\Gamma_S(R(+)M)$.

In the light of Lemma 2.11, we are going to interpret the graph $\Gamma_S(R(+)M)$ more precisely.

**Theorem 2.12.** For an $R$-module $M$, we have the following statements.

(a) If $2 \notin S$, then $\Gamma_S(R(+)M) = \bigoplus_{|M|^2} \Gamma_S(R)$.

(b) If $2 \in S$, then $\Gamma_S(R(+)M) = \bigoplus_{|M|^2} \Gamma_S(R) \oplus (\bigoplus_{|S|} K_{|M|})$.

**Proof.** Lemma 2.11 shows that each edge of $\Gamma_S(R)$ corresponds to exactly $|M|^2$ edges in $\Gamma_S(R(+)M)$. Now, if $2 \notin S$, then, for each $r \in R$, $2r \notin S$, and so, in view of Lemma 2.11, there is no adjacency between elements of $A_r$ in $\Gamma_S(R(+)M)$. Thus $\Gamma_S(R(+)M) = \bigoplus_{|M|^2} \Gamma_S(R)$. Also, if $2 \in S$, then $2r \in S$ for every $r \in S$. Hence, in view of Lemma 2.11, $A_r$ is a complete subgraph of $\Gamma_S(R(+)M)$, and so $\Gamma_S(R(+)M) = (\bigoplus_{|M|^2} \Gamma_S(R)) \oplus (\bigoplus_{|S|} K_{|M|})$. \hfill $\square$

In a similar way as we discussed in Remark 2.10, one can see that Theorem 2.12 is a slight generalization of [6, Proposition 2.9].

**Proposition 2.13.** Let $S$ be a saturated multiplicatively closed subset of $R$ with $R \setminus S = \bigcup_{i=1}^n p_i$ such that $|R/p_i| = 2$ for some $i$. Then $\Gamma_S(R)$ is a bipartite graph. Furthermore, $\Gamma_S(R)$ is a complete bipartite graph if and only if $n = 1$.

**Proof.** Since $|R/p_i| = 2$, we have that $R = p_i \cup (p_i - 1) = p_i \cup (p_i + 1)$. Set $V_1 := p_i$ and $V_2 := p_i + 1$. It is easy to see that no pair of the elements of $V_1$ are adjacent. If $x$ and $y$ in $V_2$ are adjacent, then there exist $p$ and $p'$ in $p_i$ such that $x = p + 1$ and $y = p' - 1$. Since $x + y \in S$, we have that $p + p' \in S$, which is impossible. Therefore, no pair of the elements of $V_2$ are adjacent, and so $\Gamma_S(R)$ is bipartite. If $n = 1$, then the sum of elements in $p_i$ with elements of $R \setminus p_i$ are not in $p_i$, and so they are in $S$. This means that $\Gamma_S(R)$ is complete bipartite graph.
We now prove the “only if” direction. To do this, assume that Γₜ(R) is a complete bipartite. To the contrary, assume that n > a₁x exists at least two distinct elements therefore each element of Iₚ. Hence, for some i, a₁n ∈ p₁, which is a contradiction, because p₁ is a prime ideal. Also, whenever x ∈ pⱼ, for some j, then a₁ ∈ pⱼ, which is impossible. Therefore x ∈ S, which implies that the vertices a₁ and a₂ ⋯ aₙ in V₁ are adjacent, which is the required contradiction. □

As pointed out by the referee, in Proposition 2.13, the hypothesis that |R/pᵢ| = 2 for some i, is not used in the “only if” direction.

Lemma 2.14. If I ≠ 0, then gr(Γₛ(R)) ≤ 4.

Proof. For each p ∈ I and s ∈ S, we have that p + s ∉ ∪ᵢ∈A pᵢ, and so p + s ∈ S. Therefore each element of I is adjacent to each element of S. Since I ≠ 0, there exist at least two distinct elements x and y in I. Hence, for s ∈ S, we have that deg(s) ≥ 2. Therefore, in view of Lemma 1.3, |S| ≥ 2. This means that for every two distinct vertices s₁ and s₂ in S, s₁ → x → s₂ → y → s₁ is a cycle in Γₛ(R) which implies that gr(Γₛ(R)) ≤ 4. □

In the following, we will study the possible integers appearing as the girth of the graph Γₛ(R), and we observe that these are just the integers 3, 4, 6, ∞.

Theorem 2.15. Let R be finite and S be a saturated multiplicatively closed subset of R. Then gr(Γₛ(R)) ∈ {3, 4, 6, ∞}.

Proof. Put R \ S := ∪ᵢ=₁ⁿ pᵢ. If I ≠ 0, then, by Lemma 2.14, we have gr(Γₛ(R)) ≤ 4.

Now, assume that I = 0. Since R is Artinian, each prime ideal is maximal. Therefore, by the Chinese-Remainder Theorem, the homomorphism ϕ : R → R/p₁ × R/p₂ × ⋯ × R/pₙ, with ϕ(x) = (x + p₁, ⋯ , x + pₙ) is an isomorphism, (see [7, Theorem 1.10]). Note that, for arbitrary element x ∈ R, x ∈ S if and only if for each 1 ≤ i ≤ n, x /∈ pᵢ, which is equivalent to ϕ(x) /∈ Z(ϕ(R)). Moreover, ϕ(x) /∉ Z(ϕ(R)) if and only if ϕ(x) ∈ U(ϕ(R)), which is equivalent to x ∈ U(R). Thus S = U(R). Now, the result immediately follows from [6, Proposition 5.10]. □

In the following we provide some examples of saturated multiplicatively closed sets, to show that each numbers 3, 4, 6 and ∞ given in Theorem 2.15 can appear as the girth of some graphs.

Example 2.16. Let R = Z₆. As we see in Example 1.1, gr(Γₜ(Z₆(R))) = 3, gr(Γₜ(U(R))(R)) = 6 and gr(Γₛ(R)) = 4, where S = {1, 3, 5}. For the saturated multiplicatively closed subset S = {−1, 1} of Z, one can easily show that the graph Γₛ(R) is a path and so gr(Γₛ(R)) = ∞.
In the next theorem, we characterize the rings $R$ with a saturated multiplicatively closed set $S$ such that the graph $\Gamma_S(R)$ has infinite girth, or equivalently, $\Gamma_S(R)$ is a forest.

**Theorem 2.17.** Let $R$ be finite and $S$ be a saturated multiplicatively closed subset of $R$. Then $\text{gr}(\Gamma_S(R)) = \infty$ if and only if one of the following statements holds:

(a) $R = \mathbb{Z}_3$.
(b) $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ and $|S| = 1$.

**Proof.** Whenever $R = \mathbb{Z}_3$, since $U(R) \subseteq S \neq R$, we have that $S = \{1, 2\}$. Then the graph $\Gamma_S(R)$ is a path $1 \to 0 \to 2$, and so $\text{gr}(\Gamma_S(R)) = \infty$.

Also, when $\text{char}(R) = 2$ and $|S| = 1$, then for each $a \in R$, $1 - a \neq a$. Moreover, the vertices $a$ and $1 - a$ are adjacent for all $a \in R$, which implies that, in this case, $\Gamma_S(R)$ forms a perfect matching, and so $\text{gr}(\Gamma_S(R)) = \infty$.

For a saturated multiplicatively closed subset $S$ of $R$, we have $U(R) \subseteq S$. Therefore $\text{gr}(\Gamma_S(R)) \leq \text{gr}(\Gamma_U(R)(R))$. Hence, by [6, Proposition 5.10], it is enough to study $\text{gr}(\Gamma_S(R))$ in the case that $R = \mathbb{Z}_3$ or $R = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

Note that these rings have unit graphs with infinite girth. As we have shown above, it is clear in this case that $R = \mathbb{Z}_3$. So we may assume that $R \neq \mathbb{Z}_3$. Assume $1 \neq s = (s_1, \ldots, s_n)$ is an element of $S$. Note that, by our assumption, $s \neq 0$. Set $s' := (1 + s_1, \ldots, 1 + s_n)$. Then $s' + s = 1 \in S$ and $1 + s' = s \in S$, which implies that

$$s \to 0 \to 1 \to s' \to s$$

is a cycle in $\Gamma_S(R)$. Thus $S = \{1\}$ and $\text{char}(R) = 2$, which complete the proof. \qed

By using the proof of Theorem 2.17, one can obtain the following characterization for a ring $R$ with saturated multiplicatively closed set $S$ such that the graph $\Gamma_S(R)$ is forest.

**Corollary 2.18.** Let $R$ be a finite ring such that $R \neq \mathbb{Z}_3$. Also, suppose that $S$ is a saturated multiplicatively closed subset of $R$. Then $\Gamma_S(R)$ is forest if and only if it is a complete matching.

In some special cases, we will find a better upper bound for the girth of the graph $\Gamma_S(R)$. In the following theorem, we study the local case.

**Theorem 2.19.** Suppose that $(R, \mathfrak{m})$ is a local ring and that $R \setminus S = \bigcup_{i \in A} \mathfrak{p}_i$. Then

(a) If $|A| \geq 2$, then $\text{gr}(\Gamma_S(R)) = 3$.
(b) If $|A| = 1$, then $\text{gr}(\Gamma_S(R)) \leq 4$.

**Proof.** (a) Note that $\bigcup_{i \in A} \mathfrak{p}_i \subset \mathfrak{m}$ since the equality $\bigcup_{i \in A} \mathfrak{p}_i = \mathfrak{m}$ implies $S = R \setminus \mathfrak{m}$, which is a contradiction. Assume that $x \in \mathfrak{m} \setminus \bigcup_{i \in A} \mathfrak{p}_i$. Hence $1 + x \in U(R) \subseteq S$. So $0 \to 1 \to x \to 0$ is a cycle in $\Gamma_S(R)$.

(b) If $|A| = 1$, then $S = R \setminus \mathfrak{p}$ and $I = \mathfrak{p} \neq 0$. Therefore, by Lemma 2.14, we have that $\text{gr}(\Gamma_S(R)) \leq 4$. \qed

**Lemma 2.20.** Let $(R, \mathfrak{m})$ be a finite local ring and $S$ a saturated multiplicatively closed subset of $R$. Then $S = U(R)$
Proof. Since $S$ is a saturated multiplicatively closed subset of $R$, $R \setminus S = \bigcup_{i \in A} p_i$ for some prime ideals $p_i$ of $R$. Since $R$ is a finite local ring, we have that $\text{Spec}(R) = \text{Max}(R) = \{ m \}$. So $R \setminus S = m$. On the other hand, $Z(R) = m$. Thus $S = U(R)$. □

Let $R$ be a finite ring. We can write $R = R_1 \times R_2 \times \cdots \times R_k$ such that every $R_i$ is a finite local ring with maximal ideal $m_i$. Now, let $S$ be a saturated multiplicatively closed subset of $R$. It is not hard to see that $S = S_1 \times S_2 \times \cdots \times S_k$, where, for $1 \leq i \leq k$, $S_i = \{ s_i \in R_i : (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_k) \in S \}$ is a saturated multiplicatively closed subset of $R_i$ or $S_i = R_i$.

**Proposition 2.21.** Let $R = R_1 \times R_2 \times \cdots \times R_k$, where $(R_i, m_i)$ is a finite local ring such that $R_i/m_i \cong \mathbb{Z}_2$, and let $S = S_1 \times S_2 \times \cdots \times S_k$ be a saturated multiplicatively closed subset of $R$. Then $\text{diam}(\Gamma_S(R)) \in \{ 1, 2, \infty \}$

Proof. By Lemma 2.20, we have that $S_i = U(R_i)$ or $S_i = R_i$ for each $1 \leq i \leq k$. Now, we can consider the following cases:

Case 1. Suppose that there exists $1 \leq i \neq j \leq n$ such that $S_i = U(R_i)$ and $S_j = U(R_j)$, and consider two elements $(s_1, s_2, \ldots, s_k)$ and $(a_1, a_2, \ldots, a_k)$ of $R$ such that $(s_i, s_j) = (1, 1)$ and $(a_i, a_j) = (0, 1)$. If there exists a path from $(s_1, s_2, \ldots, s_k)$ to $(a_1, a_2, \ldots, a_k)$ in $\Gamma_S(R)$ as

$$(s_1, s_2, \ldots, s_k) \rightarrow (x_{11}, \ldots, x_{1k}) \rightarrow \cdots \rightarrow (x_{m1}, \ldots, x_{mk}) \rightarrow (a_1, a_2, \ldots, a_k),$$

then we have $(x_{i_\ell}, x_{j_\ell}) \in m_i \times m_j$ or $(1 + m_i) \times (1 + m_j)$ for all $\ell$. Since $(a_i, a_j) = (0, 1)$, the vertices $(x_{m1}, \ldots, x_{mk})$ and $(a_1, \ldots, a_k)$ are not adjacent, which is impossible. So $\Gamma_S(R)$ is disconnected.

Case 2. Now suppose that there is $1 \leq i \leq k$ such that $S_i = U(R_i)$. Without loss of generality, we can assume $i = 1$. Thus $R \setminus S = m_1 \times R_2 \times \cdots \times R_k$, which is a prime ideal of $R$ and $|R/(R \setminus S)| = 2$. Now, by Proposition 2.13, $\Gamma_S(R)$ is a complete bipartite graph, and so $\text{diam}(\Gamma_S(R)) = 2$.

Case 3. If $S = R_1 \times R_2 \times \cdots \times R_k$, then $\Gamma_S(R)$ is a complete graph. Therefore $\text{diam}(\Gamma_S(R)) = 1$. □

As an immediate consequence of the proof of Proposition 2.21, we have the following corollary.

**Corollary 2.22.** Let $R = R_1 \times R_2 \times \cdots \times R_k$, where $(R_i, m_i)$ is a finite local ring such that $R_i/m_i \cong \mathbb{Z}_2$, and let $S = S_1 \times S_2 \times \cdots \times S_k$ be a saturated multiplicatively closed subset of $R$. Then $\Gamma_S(R)$ is disconnected if and only if there exist $1 \leq i \neq j \leq n$ such that $S_i = U(R_i)$ and $S_j = U(R_j)$.

**Theorem 2.23.** Let $R$ be a finite ring. For a saturated multiplicatively closed subset $S$ of $R$, we have that $\text{diam}(\Gamma_S(R)) \in \{ 1, 2, 3, \infty \}$.

Proof. Let $R = R_1 \times \cdots \times R_n$, where each $R_i$ is a local ring with maximal ideal $m_i$. If $R$ does not have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient, then by [6, Lemma 5.5] we have $\text{diam}(\Gamma_{U(R_i)}(R)) \leq 3$ which implies that $\text{diam}(\Gamma_S(R)) \leq 3$, as desired. Assume that $R_1, \ldots, R_k$ each have $\mathbb{Z}_2$ as a quotient, and $\Gamma_S(R)$ is connected. If $0 \in S$, then $S = R_1 \times R_2 \times \cdots \times R_n$. Thus the graph $\Gamma_S(R)$ is complete, and so $\text{diam}(\Gamma_S(R)) = 1$. □
Now assume that $0 \notin S$. Set $R' := R_{k+1} \times \cdots \times R_n$ and $S' := \{(s_{k+1}, \ldots, s_n) : (s_1, \ldots, s_n) \in S\}$. By the proof of [6, Lemma 5.5], in the graph $\Gamma_{S'}(R')$, for each two elements $(a_{k+1}, \ldots, a_n)$ and $(b_{k+1}, \ldots, b_n)$, there exists an element $(c_{k+1}, \ldots, c_n)$ which is adjacent to both vertices $(a_{k+1}, \ldots, a_n)$ and $(b_{k+1}, \ldots, b_n)$. Suppose that there exists at most one $i$ with $1 \leq i \leq k$ such that $S_i = U(R_i)$. By Proposition 2.21, for each two elements $(a_1, \ldots, a_n), (b_1, \ldots, b_n)$, of $R$, the following cases may be considered:

Case 1. Suppose that $(a_1, \ldots, a_k) = (b_1, \ldots, b_k)$. Then the vertex $(1 - a_1, \ldots, 1 - a_k, c_{k+1}, \ldots, c_n)$ is adjacent to both vertices $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, where $(c_{k+1}, \ldots, c_n)$ is one of the vertices which is adjacent to both vertices $(a_{k+1}, \ldots, a_n)$ and $(b_{k+1}, \ldots, b_n)$ in $\Gamma_{S'}(R')$.

Case 2. Assume that $(a_1, \ldots, a_k)$ is adjacent to $(b_1, \ldots, b_k)$. Then for the vertex $(c_{k+1}, \ldots, c_n)$, which is one of the vertices adjacent to both vertices $(a_{k+1}, \ldots, a_n)$ and $(b_{k+1}, \ldots, b_n)$, and also if $(d_{k+1}, \ldots, d_n)$ is one of the vertices adjacent to both vertices $(c_{k+1}, \ldots, c_n)$ and $(b_{k+1}, \ldots, b_n)$ in $\Gamma_{S'}(R')$, we have the following path in $\Gamma_S(R)$.

$$(a_1, \ldots, a_n) \rightarrow (b_1, \ldots, b_k, c_{k+1}, \ldots, c_n) \rightarrow (a_1, \ldots, a_k, d_{k+1}, \ldots, d_n) \rightarrow (b_1, \ldots, b_n)$$

Case 3. Consider the path $(a_1, \ldots, a_k) \rightarrow (c_1, \ldots, c_k) \rightarrow (b_1, \ldots, b_k)$. Then for the vertex $(c_{k+1}, \ldots, c_n)$, which is one of the vertices adjacent to both vertices $(a_{k+1}, \ldots, a_n)$ and $(b_{k+1}, \ldots, b_n)$ in $\Gamma_{S'}(R')$, we have the following path in $\Gamma_S(R)$.

$$(a_1, \ldots, a_n) \rightarrow (c_1, \ldots, c_k, c_{k+1}, \ldots, c_n) \rightarrow (b_1, \ldots, b_n)$$

If there exist two indices $i \neq j$ with $S_i = U(R_i)$ and $S_j = U(R_j)$, then, by applying a method similar to that we used in the proof of Proposition 2.21, one can deduce that $\Gamma_S(R)$ has infinite diameter, which completes the proof.

As an immediate consequence of the proof of Theorem 2.23, we have the following characterization for disconnected graphs.

**Corollary 2.24.** Let $R = R_1 \times \cdots \times R_n$ be a finite ring and $S$ be a saturated multiplicatively closed subset of $R$. Then $\Gamma_S(R)$ is disconnected if and only if there exist $i \neq j$ where $R_i$ and $R_j$ have $\mathbb{Z}_2$ as a quotient and $S_i = U(R_i)$ and $S_j = U(R_j)$.

**Acknowledgments**
The authors are deeply grateful to the referee for careful reading of the manuscript and helpful suggestions.

**References**


Zahra Barati  
*E-mail address: za.barati87@gmail.com*

Kazem Khashyarmanesh  
*E-mail address: khashyar@ipm.ir*

Fatemeh Mohammadi  
*E-mail address: fatemeh.mohammadi716@gmail.com*

Khosro Nafar  
*E-mail address: khosronafar@yahoo.com*

Department of Pure Mathematics, Ferdowsi University of Mashhad P.O. Box 1159-91775, Mashhad, Iran.