SEQUENTIALLY COHEN-MACAULAY GRAPHS OF FORM $\theta_{n_1, \ldots, n_k}$

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Abstract. Let $k$ be an integer greater than 2 and $n_1, \ldots, n_k$ be a sequence of positive integers with at most one of them being equal to 1. Let $\theta_{n_1, \ldots, n_k}$ be a graph consisting of $k$ paths, having only their endpoints in common. We characterize all sequentially Cohen-Macaulay graphs of this type. We also show for these types of graphs the notions of vertex decomposable, shellable and sequentially Cohen-Macaulay are equivalent.

1. Introduction

Let $G$ be a finite simple graph. To $G$ with vertex set $[n] = \{1, \ldots, n\}$ and edge set $E(G)$, one can associate an ideal $I(G) \subset R = K[x_1, \ldots, x_n]$, called the edge ideal of $G$, which is generated by all monomials $x_i x_j$ such that $\{i, j\} \in E(G)$. Here, $K$ is an arbitrary field. The independence complex $\Delta_G$ of a graph $G$ is defined by

$$\Delta_G = \{A \subseteq V \mid A \text{ is an independent set in } G\},$$

where, $A$ is an independent set in $G$ if none of its elements are adjacent. Note that $\Delta_G$ is precisely the simplicial complex associated with $I(G)$.

It is a well-known consequence of Menger’s Theorem [5, Theorem 3.3.5] that each 3-connected graph has an induced subgraph of the form

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\(\theta_{p,q,r}\), for some natural numbers \(p, q\) and \(r\). This was our motivation to study sequentially Cohen-Macaulay graphs of the form \(\theta_{n_1,\ldots,n_k}\).

A graded \(R\)-module \(M\) is called \textit{sequentially Cohen-Macaulay} (over \(K\)) if there exists a finite filtration of graded \(R\)-modules,

\[0 = M_0 \subset M_1 \subset \cdots \subset M_r = M,\]

such that each \(M_i/M_{i-1}\) is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing; that is,

\[\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).\]

A graph \(G\) is said to be sequentially Cohen-Macaulay, if \(R/I(G)\) is a sequentially Cohen-Macaulay \(R\)-module.

On the other hand, a simplicial complex \(\Delta\) is called \textit{shellable}, in the sense of Björner and Wachs [1], if the facets (maximal faces) of \(\Delta\) can be ordered as \(F_1, \ldots, F_s\) such that for all \(1 \leq i < j \leq s\), there exists some \(v \in F_j \setminus F_i\) and some \(l \in \{1, \ldots, j-1\}\) with \(F_j \setminus F_l = \{v\}\). A graph \(G\) is called shellable, if \(\Delta_G\) is a shellable simplicial complex. In [12], Stanley showed that every shellable simplicial complex was sequentially Cohen-Macaulay, but the converse was not true.

Studying shellable or sequentially Cohen-Macaulay graphs has attracted significant attentions of researchers working in the borderline of combinatorial commutative algebra and algebraic combinatorics; see [1, 6, 7, 8, 10, 14, 16]. In [8], Francisco and Van Tuyl characterized all sequentially Cohen-Macaulay cycles. They showed that the \(n\)-cycle \(C_n\) was sequentially Cohen-Macaulay if and only if \(n \in \{3, 5\}\) (see [8, Proposition 4.1]). In [6], Faridi showed that simplicial trees were sequentially Cohen-Macaulay. Moreover, in [10], sequentially Cohen-Macaulay cacti graphs (a cactus is a connected graph in which each edge belongs to at most one cycle) were characterized. In addition, in [14], Van Tuyl and Villarreal showed that a bipartite graph \(G\) was shellable if and only if it was sequentially Cohen-Macaulay (see [14, Theorem 3.8]).

Here, we determine all sequentially Cohen-Macaulay graphs of the form \(\theta_{n_1,\ldots,n_k}\), where \(\{n_1, \ldots, n_k\} \neq \{2, 5\}\). For \(\{n_1, \ldots, n_k\} \neq \{2, 5\}\), we show in Theorem 2.6 that \(\theta_{n_1,\ldots,n_k}\) is sequentially Cohen-Macaulay if and only if \(\{1,2\} \subseteq \{n_1,\ldots,n_k\}\) or \(\{2,3\} \subseteq \{n_1,\ldots,n_k\}\) or \(\{n_1,\ldots,n_k\} = \{1,4\}\). Moreover, as a result of this theorem, in Theorem 2.7 we show those graphs of the form \(\theta_{n_1,\ldots,n_k}\), which satisfy each one of the latter relations, are sequentially Cohen-Macaulay if and only if they are shellable or vertex decomposable.
Finally, in Proposition 2.8, we show that for \( \{n_1, \ldots, n_k\} = \{2, 5\} \), the graph \( \theta_{n_1, \ldots, n_k} \) is not vertex decomposable. Therefore, we characterize all vertex decomposable graphs of the form \( \theta_{n_1, \ldots, n_k} \) in Theorem 2.9. In Proposition 2.10, by direct computation, we show that for \( k = 3 \) and \( \{n_1, \ldots, n_k\} = \{2, 5\} \), the graph \( \theta_{n_1, \ldots, n_k} \) is not even sequentially Cohen-Macaulay. This result and computational evidences from some other examples lead us to conjecture that all graphs of the form \( \theta_{n_1, \ldots, n_k} \), for which \( \{n_1, \ldots, n_k\} = \{2, 5\} \), are not sequentially Cohen-Macaulay.

Characterizing vertex decomposable, shellable and sequentially Cohen-Macaulay graphs of the form \( \theta_{n_1, \ldots, n_k} \) with [13, Lemma 2.4] and [14, Theorem 2.9] enable us to get more examples of vertex decomposable, shellable and sequentially Cohen-Macaulay graphs.

2. Sequentially Cohen-Macaulay graphs of the form \( \theta_{n_1, \ldots, n_k} \)

Let \( k \) be an integer greater than 1 and \( n_1, \ldots, n_k \) be a sequence of positive integers. Let \( \theta_{n_1, \ldots, n_k} \) be the graph constructed by \( k \) paths of length \( n_1, \ldots, n_k \), with only their endpoints being in common. By length of a path, we mean the number of edges in the path. Since the graphs are assumed simple, at most one of the \( n_i \)s in \( \theta_{n_1, \ldots, n_k} \) can be equal to one. If \( k = 2 \), then \( \theta_{n_1, \ldots, n_k} \) would be a cycle of length \( n_1 + n_2 \). The vertex decomposable and sequentially Cohen-Macaulay graphs of these types are completely studied in [8, 16]. Here, we assume \( k > 2 \) and characterize all vertex decomposable, shellable and sequentially Cohen-Macaulay graphs of the form \( \theta_{n_1, \ldots, n_k} \).

Given a simplicial complex \( \Delta \) on \( [n] \), the Alexander dual complex \( \Delta^\vee \) is defined by \( \Delta^\vee = \{[n] \setminus F | F \not\in \Delta\} \). Unless otherwise stated, when we discuss the Alexander dual \( \Delta^\vee \) of a simplicial complex \( \Delta \), we assume that \( [n] \setminus i \not\in \Delta \), for all \( i \in [n] \). Thus, \( \Delta^\vee \) is again a simplicial complex on \( [n] \).

Let \( I = (x_{1,1} \cdots x_{1,s_1}, \ldots, x_{t,1} \cdots x_{t,s_t}) \) be a square-free monomial ideal. The ideal

\[
I^\vee = (x_{1,1}, \ldots, x_{1,s_1}) \cap \ldots \cap (x_{t,1}, \ldots, x_{t,s_t})
\]

is called the Alexander dual of \( I \). These two ideals are related in the following way. If \( I \) is the Stanley-Reisner ideal of a simplicial complex \( \Delta \), then the Stanley-Reisner ideal of its Alexander dual \( \Delta^\vee \) is \( I^\vee \).

Another related notion is componentwise linear ideals, introduced by Herzog and Hibi, to characterize sequentially Cohen-Macaulay ideals.
Let \( I \) be a graded ideal of \( R \) and let \( I_{<d>} \) be the ideal generated by all homogeneous polynomials of degree \( d \) of \( I \). A graded ideal \( I \) of \( R \) is called \textit{componentwise linear} if \( I_{<d>} \) has a linear resolution, for every \( d \). Let \( I \) be a square-free monomial ideal in a polynomial ring. The ideal generated by the square-free monomials of degree \( d \) of \( I \) is denoted by \( I_{[d]} \). Herzog and Hibi in [9, Proposition 1.5] showed that the square-free ideal \( I \) was componentwise linear if and only if \( I_{[d]} \) had a linear resolution for every \( d \).

Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). A subset \( C \subseteq V(G) \) is a \textit{minimal vertex cover} of \( G \) if: (1) every edge of \( G \) is incident with one vertex in \( C \), and (2) there is no proper subset of \( C \) with the first property. In [8], Francisco and Van Tuyl showed that if \( \mathcal{I}(G) \) was the ideal of a graph \( G \), then

\[
\mathcal{I}(G)^\vee_{[d]} = (\{x_{i_1} \cdots x_{i_d}|\text{\{x}_{i_1}, \ldots, x_{i_d}\} \text{ is a vertex cover of } G \text{ of size } d\}).
\]

In [9], Herzog and Hibi showed the following theorem to be used in the proof of Proposition 2.4.

\textbf{Theorem A.} Let \( I \) be a square-free monomial ideal in a polynomial ring. Then \( I^\vee \) is componentwise linear if and only if \( R/I \) is sequentially Cohen-Macaulay.

Let \( N(v) \) be the set of all adjacent vertices of \( v \) and let \( N[v] = N(v) \cup \{v\} \). Vertex decomposability was introduced by Provan and Billera [11] in the pure case, and extended to the non-pure case by Björner and Wachs [2]. We will use the following definition of vertex decomposable graphs which is an interpretation of the definition of vertex decomposable for the independence complex of a graph, as stated in [13, 16].

\textbf{Definition 2.1.} The independence complex of \( G \) is vertex decomposable if \( G \) is a totally disconnected graph (with no edges), or if

- \( G \setminus v \) and \( G \setminus N[v] \) are both vertex decomposable, and
- No independent set in \( G \setminus N[v] \) is a maximal independent set in \( G \setminus v \).

A vertex \( v \) which satisfies in these conditions is called a shedding vertex.
The graph $G$ is called vertex decomposable if its independence complex is vertex decomposable. It is known that the any vertex decomposable graph is shellable and so is sequentially Cohen-Macaulay (see [16]).

For characterizing vertex decomposable, shellable and sequentially Cohen-Macaulay graphs of the form $\theta_{n_1,\ldots,n_k}$, we have to distinguish among some cases, depending on $n_1,\ldots,n_k$, as follows.

**Proposition 2.2.** If $\{1,2\} \subseteq \{n_1,\ldots,n_k\}$, then $\theta_{n_1,\ldots,n_k}$ is vertex decomposable and so is shellable and sequentially Cohen-Macaulay.

**Proof.** Two paths of length one and two form a triangle. Let $v,u$ and $w$ be its vertices such that $\deg(v) = 2$. The graphs $\theta_{n_1,\ldots,n_k} \setminus \{u\}$ and $\theta_{n_1,\ldots,n_k} \setminus N[u]$ are chordal and so they are vertex decomposable, by [16, Theorem 1]. For any independent set $F$ in $\theta_{n_1,\ldots,n_k} \setminus \{u\}$, $F \cup \{v\}$ is an independent set in $\theta_{n_1,\ldots,n_k} \setminus \{u\}$. Therefore, $\theta_{n_1,\ldots,n_k}$ fulfills the conditions of Definition 2.1, which completes the proof. □

**Remark 2.3.** If in the above proposition, one assumes $\{n_1,\ldots,n_k\} = \{1,2\}$, then the associated graph, $\theta_{n_1,\ldots,n_k}$, is chordal. These types of graphs are known to be vertex decomposable, by [16, Theorem 1].

A chordless path in a graph $G$ is a path $v_1, v_2, \ldots, v_k$ in $G$ with no edge $v_iv_j$ with $j \neq i + 1$. A simplicial $k$-path in $G$ is a chordless path $v_1, v_2, \ldots, v_k$ which cannot be extended on both endpoints to a chordless path $v_0, v_1, \ldots, v_k, v_{k+1}$ in $G$.

**Proposition 2.4.** Let $\{2,3\} \subseteq \{n_1,\ldots,n_k\}$. Then, $\theta_{n_1,\ldots,n_k}$ is vertex decomposable and consequently shellable and sequentially Cohen-Macaulay.

**Proof.** Let $P_1 : u, x, v$ and $P_2 : u, y, z, v$ be two paths of length two and three in $\theta_{n_1,\ldots,n_k}$. Since the path $P : x, u, y$ is a simplicial 3-path, which is not a subgraph of any chordless $C_4$, by [16, Lemma 4.3] we deduce that $G$ is vertex decomposable. □

**Proposition 2.5.** Let $\{n_1,\ldots,n_k\} = \{1,4\}$. Then, $\theta_{n_1,\ldots,n_k}$ is vertex decomposable and consequently shellable and sequentially Cohen-Macaulay.
Proof. Each cycle other than $C_5$ in $\theta_{n_1, \ldots, n_k}$ has a chord and so, by [16, Theorem 1], it is vertex decomposable. □

The following theorem is one of the main results of this paper which characterizes all sequentially Cohen-Macaulay graphs of the form $\theta_{n_1, \ldots, n_k}$, where \{n_1, \ldots, n_k\} \neq \{2, 5\}.

Theorem 2.6. Let $n_1, \ldots, n_k \neq \{2, 5\}$. Then, $\theta_{n_1, \ldots, n_k}$ is sequentially Cohen-Macaulay if and only if one of the following holds:

1. \{1, 2\} \subseteq \{n_1, \ldots, n_k\}.
2. \{2, 3\} \subseteq \{n_1, \ldots, n_k\}.
3. \{1, 4\} = \{n_1, \ldots, n_k\}.

Proof. "If". Suppose that one of (1) to (3) holds. Then, by Proposition 2.2, Proposition 2.4 and Proposition 2.5, the result holds.

"Only if". Let $G = \theta_{n_1, \ldots, n_k}$ be a sequentially Cohen-Macaulay graph. The proof is by induction on $k$. If $k = 2$, then the graph is a cycle and so the result holds by [8, Proposition 4.1]. Let $k > 2$, $n_1 \leq \cdots \leq n_k$ and $P_i : x, x_i, 1, \ldots, x_i, n_i - 1, y$, for $1 \leq i \leq k$, be the paths which construct $G$. If $n_t \geq 6$, for some $t \geq 3$, then

$$H = G \setminus \bigcup_{i=t}^{k} (N[x_{i,2}] \cup N[x_{i,n_i-2}])$$

has a component of the form $\theta_{n_1, \ldots, n_{t-1}}$. So, by the induction hypothesis, (1) or (2) or (3) holds, for $\theta_{n_1, \ldots, n_{t-1}}$. If (1) or (2) holds for $\theta_{n_1, \ldots, n_{t-1}}$, then this holds, for $\theta_{n_1, \ldots, n_k}$. Let (3) holds for $\theta_{n_1, \ldots, n_{t-1}}$, but \{n_1, \ldots, n_k\} \neq \{1, 4\}. Let $S = \{j; n_j = 4\}$ and $H' = G \setminus \bigcup_{j \in S} N[x_{j,2}]$. Since $n_2 = 4$, then $H'$ has no path of length two, three and four. By the induction hypothesis, $H'$ is not sequentially Cohen-Macaulay, which is a contradiction by [14, Theorem 3.3].

So, we can assume that $n_k < 6$. Since $G$ has no vertex of degree one, it is not a bipartite graph by [14, Lemma 2.8]. Therefore, for $n_k = 2$, we have $n_i = 1$ and so (1) holds. Similarly, if $n_k = 3$, then $n_i = 2$, for some $i$, and so (2) holds. If $n_k = 4$, then $G \setminus N[x_{k,2}]$ is $\theta_{n_1, \ldots, n_{k-1}}$. If (1), (2) or (3) holds, for $\theta_{n_1, \ldots, n_{k-1}}$, then the similar statement holds for $G$. So, assume that $n_k = 5$. Since $G$ is not bipartite, for some $i$ we have $n_i = 2$ or 4. If $n_i = 4$ for some $i$, then $H = G \setminus N[x_{i,2}]$ is sequentially Cohen-Macaulay and so (1) or (2) holds, which completes the result.
Sequentially Cohen-Macaulay graphs of form $\theta_{n_1, \ldots, n_k}$

Otherwise, the assumption $\{n_1, \ldots, n_k\} \neq \{2, 5\}$ shows that $n_j = 1$ or 3, for some $j$, and so (1) or (2) holds.

Recently, Van Tuyl showed that in bipartite graphs, the three concepts vertex decomposability, shellability and sequentially Cohen-Macaulayness are equivalent; see [13, Theorem 2.10]. Using the proof of the above theorem, we have the same property for $\theta_{n_1, \ldots, n_k}$, where $\{n_1, \ldots, n_k\} \neq \{2, 5\}$.

Theorem 2.7. Let $n_1, \ldots, n_k \neq \{2, 5\}$. Then, the followings are equivalent:

(i) $\theta_{n_1, \ldots, n_k}$ is sequentially Cohen-Macaulay.
(ii) $\theta_{n_1, \ldots, n_k}$ is shellable.
(iii) $\theta_{n_1, \ldots, n_k}$ is vertex decomposable.

Proof. Note that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) always holds for any graph. It is enough to show that for these type of graphs, (i) $\Rightarrow$ (iii). Let $\theta_{n_1, \ldots, n_k}$ be a sequentially Cohen-Macaulay graph. Then, Theorem 2.6 shows that $\theta_{n_1, \ldots, n_k}$ satisfies one of the relations of Theorem 2.6. Therefore, by Proposition 2.2, Proposition 2.4 and Proposition 2.5, we deduce that $\theta_{n_1, \ldots, n_k}$ is vertex decomposable.

In the following, we consider the case $\{n_1, \ldots, n_k\} = \{2, 5\}$.

Proposition 2.8. Let $\{n_1, \ldots, n_k\} = \{2, 5\}$. Then, $\theta_{n_1, \ldots, n_k}$ is not vertex decomposable.

Proof. Let $P_1, \ldots, P_s$ be the paths of length two in $G = \theta_{n_1, \ldots, n_k}$ and $P_{s+1}, \ldots, P_k$ be the paths of length five in $G$. Consider the labeling for $G$ such that $P_j : u, \alpha_j, v$, for $1 \leq j \leq s$, and $P_j : u, x_{j,1}, x_{j,2}, x_{j,3}, x_{j,4}, v$, for $s+1 \leq j \leq k$. We claim that no vertex of $G$ is a shedding vertex to deduce that $G$ is not vertex decomposable. For any $s+1 \leq j \leq k$, the independent set $\{u, x_{s+1,4}, \ldots, x_{k,4}\}$ is maximal in both graphs $G \setminus x_{j,2}$ and $G \setminus N[x_{j,2}]$. For the other vertices of $G$, the similar arguments hold. Therefore, $G$ is not vertex decomposable.

Proposition 2.8 and Theorem 2.6 imply the following characterization of the vertex decomposable graphs of the form $\theta_{n_1, \ldots, n_k}$.

Theorem 2.9. Let $n_1, \ldots, n_k$ be a sequence of positive integers. Then, $\theta_{n_1, \ldots, n_k}$ is vertex decomposable if and only if one of the followings holds:
The next result extends Proposition 2.8 to show that for \( k = 3 \), those graphs are not even sequentially Cohen-Macaulay.

**Proposition 2.10.** The graphs \( \theta_{2,2,5} \) and \( \theta_{2,5,5} \) are not sequentially Cohen-Macaulay.

**Proof.** Consider the labeling for \( \theta_{2,2,5} \) and \( \theta_{2,5,5} \) as given in Figure 1 and Figure 2. By [8, Lemma 2.3], the minimal generators of \( I(\theta_{2,2,5})^\vee \), correspond to the minimal vertex covers of \( \theta_{2,2,5} \) and these minimal vertex covers correspond precisely to minimal prime ideals of \( I(\theta_{2,2,5}) \). Therefore, by finding the minimal prime ideals of \( I(\theta_{2,2,5}) \), the monomials

\[
\begin{align*}
&x_1x_2x_4x_6, x_1x_3x_4x_6, x_2x_4x_6x_7x_8, x_1x_3x_5x_6, x_2x_4x_5x_7x_8, x_2x_3x_5x_7x_8, \quad \text{and} \\
&x_1x_3x_5x_7x_8,
\end{align*}
\]

generate the ideal \( I(\theta_{2,2,5})^\vee \). With computation by CoCoA, we see that \( I(\theta_{2,2,5})_{[5]} \) has the minimal graded free resolution as:

\[
0 \to R^3(-8) \to R^{12}(-7)(+)R(-8) \to R^{23}(-6) \to R^{14}(-5) \to R.
\]

Thus, it does not have a linear resolution. Therefore, \( \theta_{2,2,5} \) is not sequentially Cohen-Macaulay, by Theorem A.

Similarly, the minimal prime ideals of \( I(\theta_{2,5,5}) \) generate the ideal \( I(\theta_{2,5,5})^\vee \). By computation, we deduce that \( I(\theta_{2,5,5})_{[7]} \) has the minimal graded free resolution as:

\[
\cdots \to R^{55}(-10)(+)R(-11) \to R^{121}(-9) \to R^{124}(-8) \to R^{49}(-7) \to R.
\]

Thus, \( I(\theta_{2,5,5})_{[7]} \) does not have a linear resolution and so \( I(\theta_{2,5,5})^\vee \) is not componentwise linear. Therefore, \( \theta_{2,5,5} \) is not sequentially Cohen-Macaulay by Theorem A.

\[
\begin{align*}
\text{Figure 1} & \quad \text{Figure 2}
\end{align*}
\]
In view of Proposition 2.8 and Proposition 2.10, we conjecture that the answer to the following questions is positive.

**Question 2.11.** Let $K > 2$ and $\{n_1, \ldots, n_k\} = \{2, 5\}$. Is $\theta_{n_1, \ldots, n_k}$ not shellable? Is $\theta_{n_1, \ldots, n_k}$ not sequentially Cohen-Macaulay?

Theorem 2.6 with [14, Theorem 2.9] enable us to get more examples of shellable and sequentially Cohen-Macaulay graphs.

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