Many Triangulated 3-spheres

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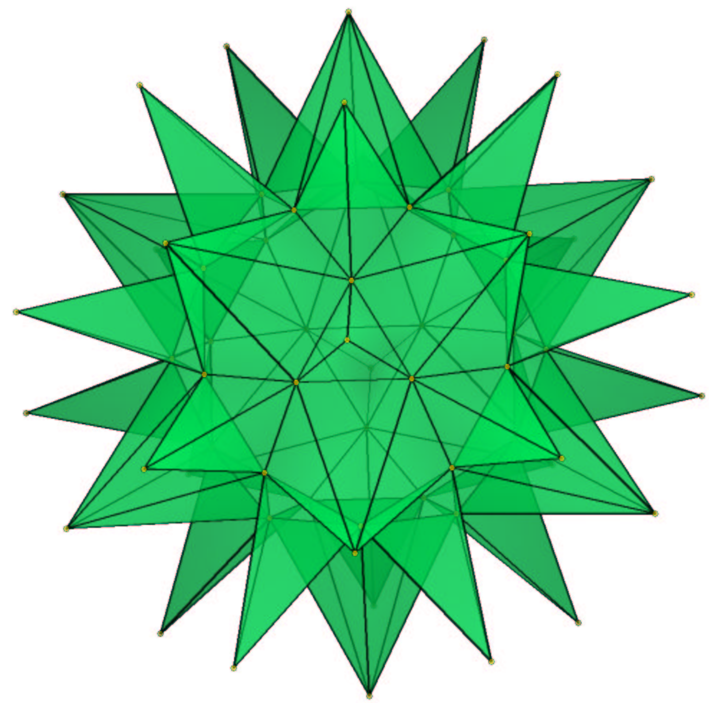
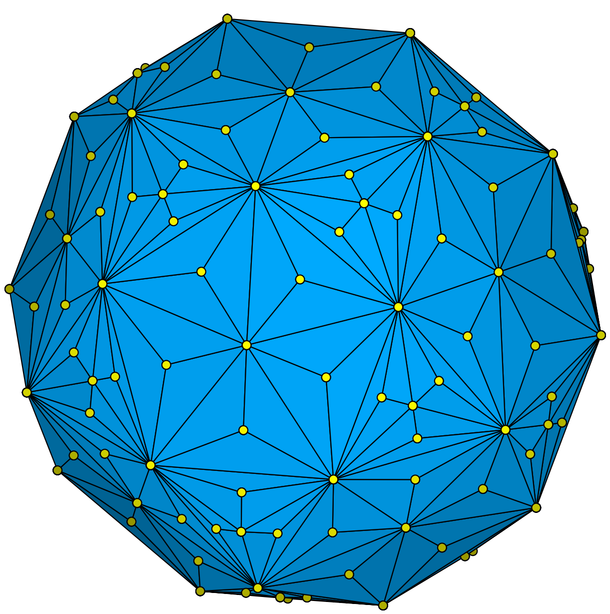
Polytopes and Spheres



A 2-dimensional **triangulated sphere** is. . .

- a simplicial complex whose underlying space is homeomorphic to S^2



. . . **for example**,
the boundary complex of a 3-polytope

$\Longrightarrow$ **"Realization"** of the sphere

analogous in higher dimensions

Are all spheres boundaries of polytopes?

- **Theorem** (reformulation of [Steinitz 1922])

  Every 2-sphere arises as the boundary complex of some 3-polytope

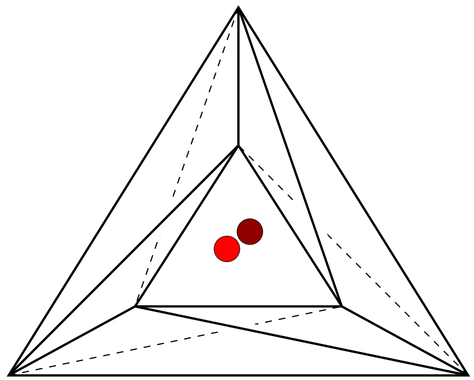
  $\Longrightarrow$ *"All 2-dimensional spheres are polytopal"*

- **Sporadic examples** of non-polytopal, simplicial 3-spheres:

  – Brückner's sphere on 8 vertices; first example

  – Barnette's sphere on 8 vertices

  – . . .



3-dimensional spheres

- In dimension 4:
  - There are $2^{O(n \log n)}$ 4-polytopes [Goodman & Pollack 1986]
  - Kalai constructed $2^{\Omega(n)}$ 3-spheres [Kalai 1988]

- **Theorem.** [P 2001] **All** of Kalai's 3-spheres are polytopal.

- **Theorem.** [P & Ziegler, 2002] There are

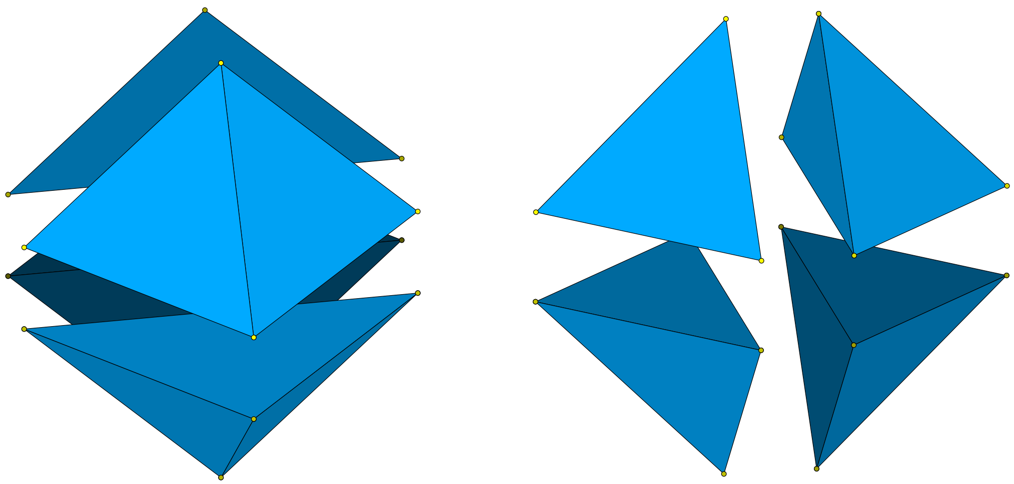
$$2^{\Omega(n^{5/4})} \gg 2^{O(n \log n)}$$

triangulated 3-dimensional spheres on n vertices.

Towards many 3-spheres

- Construct a 3-dimensional cellular sphere with
  - n vertices
  - $\Omega(n^c)$ octahedral facets, where $c > 1$
- Subdivide each octahedral facet in 2 ways:

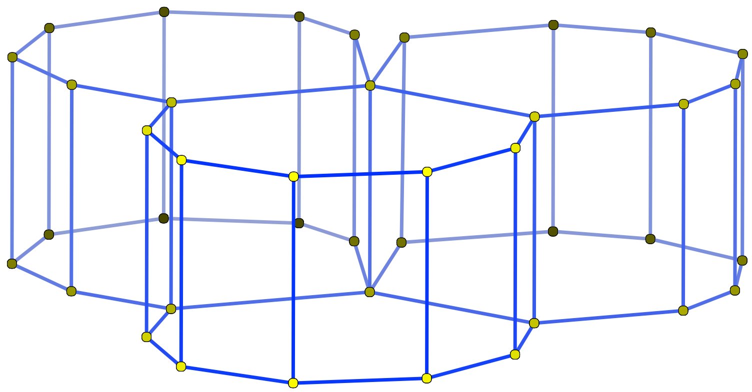
$$2^{\Omega(n^c)} \quad \textbf{3-spheres}$$

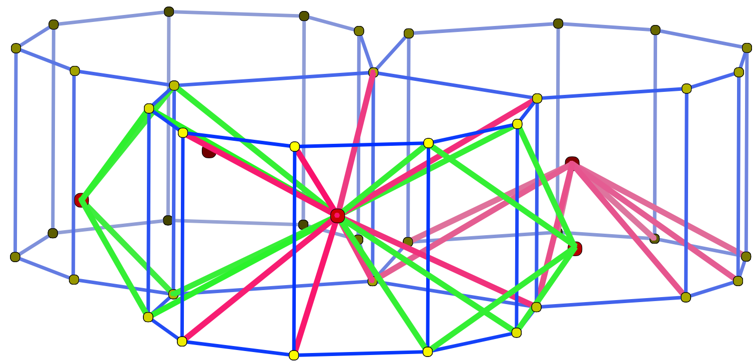


- *Most* of them are distinct: Only $\leq n!$ combinatorial automorphisms, and

$$n! \;=\; 2^{\Theta(n \log n)} \;\ll\; 2^{\Omega(n^c)}$$

The E-Construction [EKZ 2002]





$$f(\cdot) = (f_0,\ f_1,\ f_2,\ f_3) \implies f(E(\cdot)) = (f_0 + f_3,\ \cdot,\ \cdot,\ f_2)$$

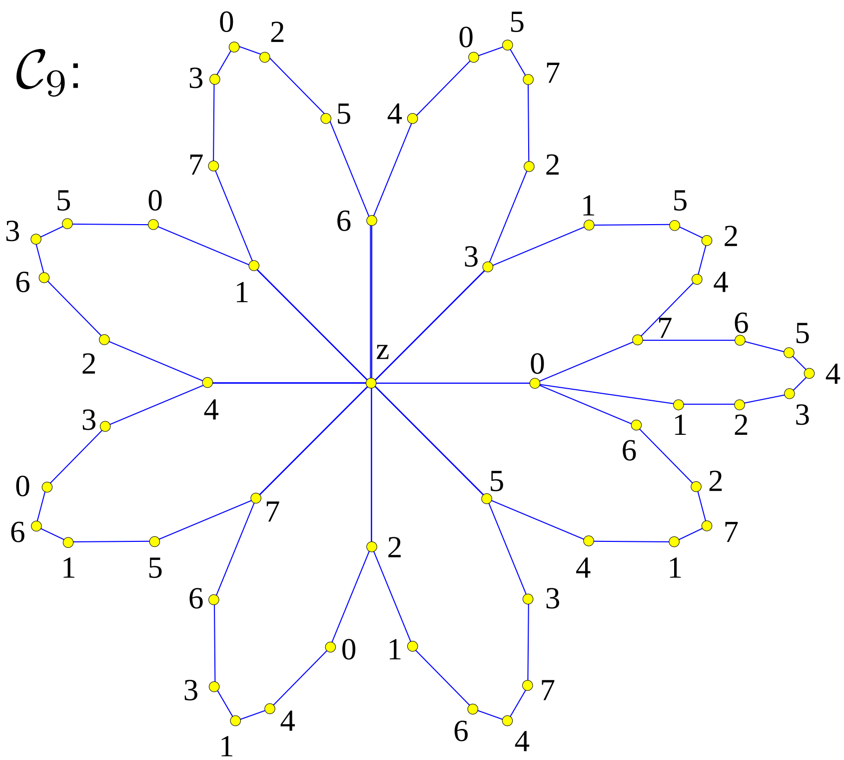
$$= \big(\boldsymbol{n},\ \boldsymbol{\Theta(n^c)},\ \boldsymbol{\Theta(n^c)},\ \boldsymbol{\Theta(n)}\big) \qquad = \big(\boldsymbol{\Theta(n)},\ \cdot,\ \cdot,\ \boldsymbol{\Theta(n^c)}\big)$$

$\Theta(n^c)$ of the 3-cells are octahedra!

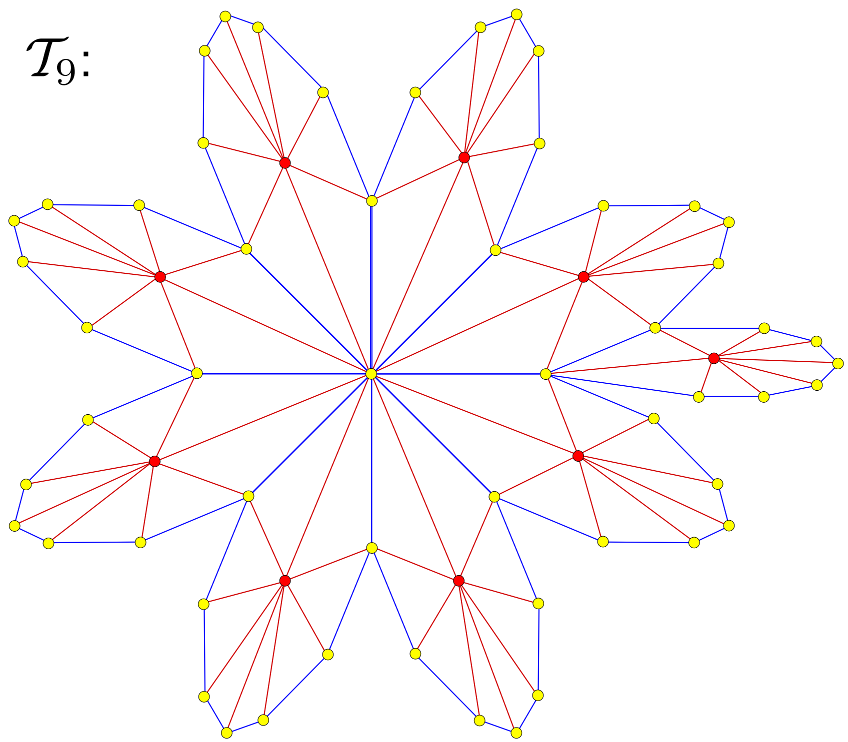
triangulate the rest . . .

Heffter's [1898] embedding of the complete graph

$\mathcal{C}_9$:

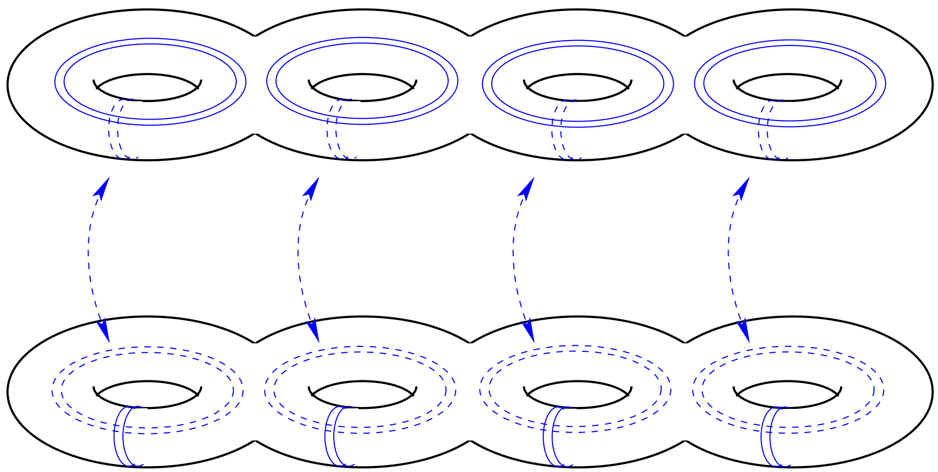


$\mathcal{T}_9$:



- Uses arithmetic in the finite field $\mathbb{F}_q$ (here: $q = 9$)
- Yields cellulation $\mathcal{C}_q$ of the surface S_g of genus $g = q(q-5)/4 + 1$
- This cellulation $\mathcal{C}_q$ has f-vector $\left(q, \binom{q}{2}, q\right)$

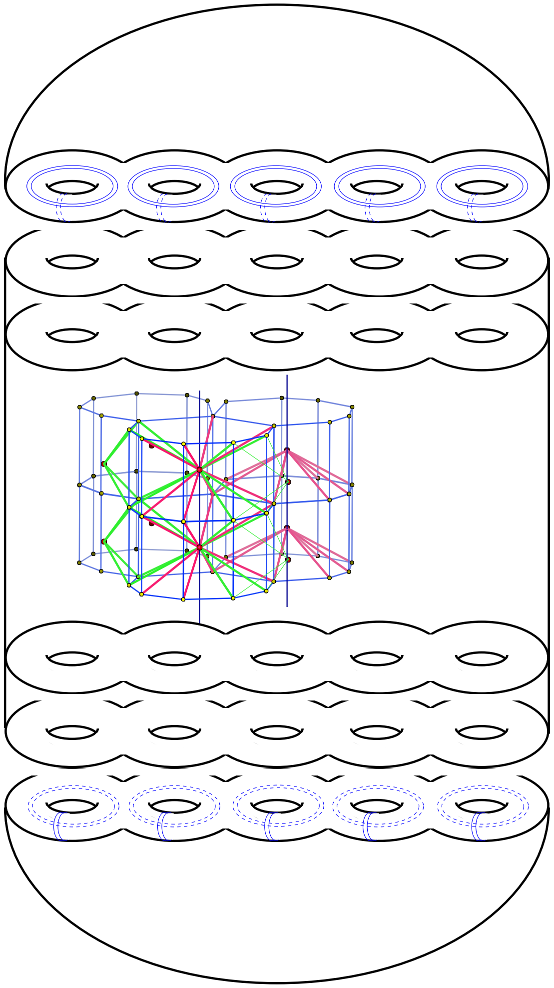
Heegaard splittings



Theorem. For any $g \geq 1$, the 3-sphere may be decomposed into two solid handlebodies $S^3 = H_1 \cup H_2$ that are identified along a surface $S_g = H_1 \cap H_2$ of genus g.

Many triangulated 3-spheres

- Start with Heegaard splitting of S^3 of genus g:
$$S^3 \;=\; H_1 \cup H_2, \quad H_1 \cap H_2 \;=\; S_g$$
- Draw Heffter triangulation $\mathcal{T}_q$ on S_g
- Thicken boundary:
$$S_g \mapsto S_g \times I, \quad \mathcal{T}_q \mapsto \mathcal{T}_q \times I$$
- **Triangulate H_1 and H_2:**
$$f(H_i) = \big(O(\boldsymbol{q}^4), O(\boldsymbol{q}^4), O(\boldsymbol{q}^4), O(\boldsymbol{q}^4)\big)$$
- Subdivide the interval I into $\boldsymbol{\Theta(q^3)}$ pieces and apply E-construction in stack of prisms
$$f(\mathcal{C}') = \big(\Theta(\boldsymbol{q}^4), \Theta(\boldsymbol{q}^5), \Theta(\boldsymbol{q}^5), \Theta(\boldsymbol{q}^5)\big)$$



Open questions

- Improve the bounds: $S(3,n) \gg 2^{\Omega(n^c)}, \quad 5/4 \leq c \leq 2$
- Show non-polytopality directly
- Are these spheres e.g. shellable?