

# Many Triangulated 3-spheres

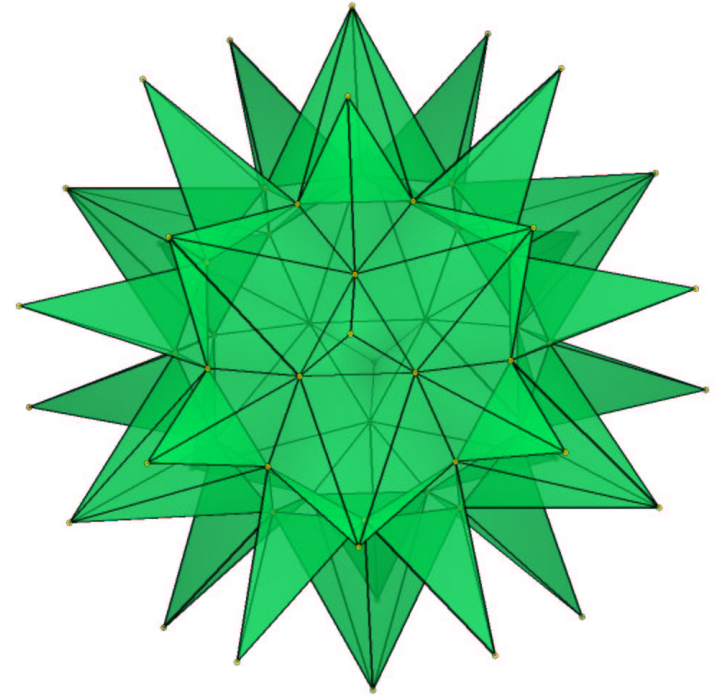
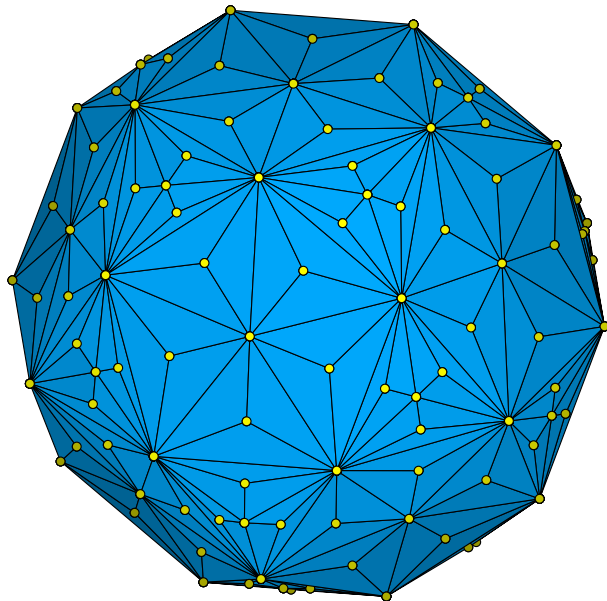
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7<sup>th</sup> BADMath Day,  
SFSU, 2003

# Polytopes and Spheres

A 2-dimensional **triangulated sphere** is. . .

- ▶ a simplicial complex whose underlying space is homeomorphic to  $S^2$



. . . **for example**,  
the boundary complex of a 3-polytope

⇒ **“Realization”** of the sphere

analogous in higher dimensions

# Are all spheres boundaries of polytopes?

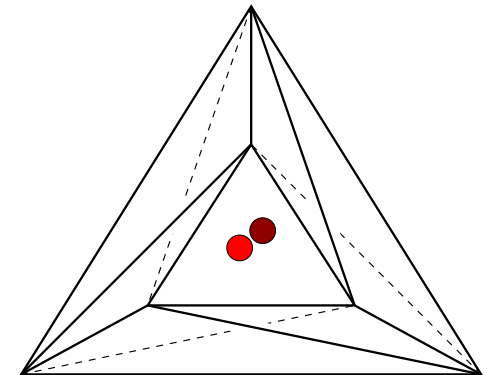
▶ **Theorem** (reformulation of [Steinitz 1922])

Every 2-sphere arises as the boundary complex of some 3-polytope

$\implies$  “All 2-dimensional spheres are polytopal”

▶ **Sporadic examples** of non-polytopal, simplicial 3-spheres:

- Brückner’s sphere on 8 vertices; first example
- Barnette’s sphere on 8 vertices
- . . .



## 3-dimensional spheres

▶ In dimension 4:

- There are  $2^{O(n \log n)}$  4-polytopes [Goodman & Pollack 1986]
- Kalai constructed  $2^{\Omega(n)}$  3-spheres [Kalai 1988]

▶ **Theorem.** [P 2001] **All** of Kalai's 3-spheres are polytopal.

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▶ **Theorem.** [P & Ziegler, 2002] There are

$$2^{\Omega(n^{5/4})} \gg 2^{O(n \log n)}$$

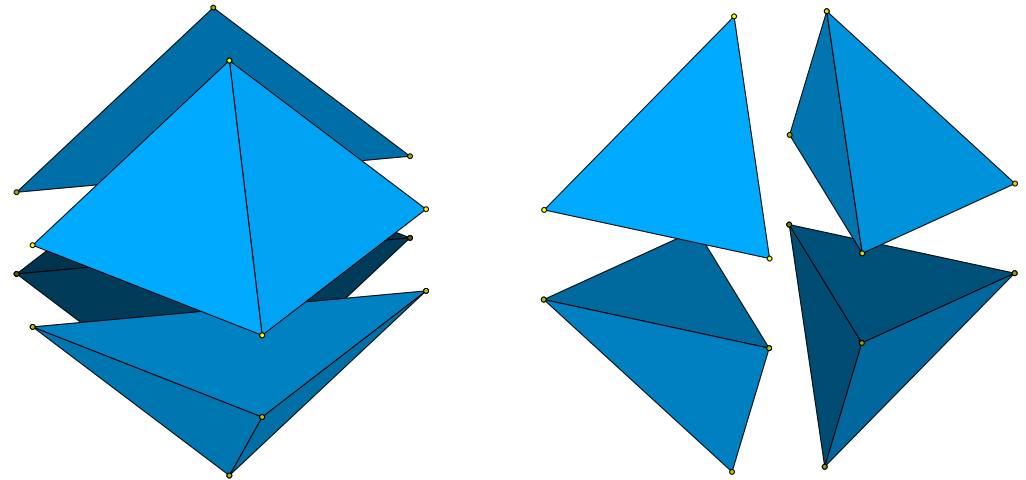
triangulated 3-dimensional spheres on  $n$  vertices.

## Towards many 3-spheres

- ▶ Construct a 3-dimensional cellular sphere with
  - $n$  vertices
  - $\Omega(n^c)$  octahedral facets, where  $c > 1$

- ▶ Subdivide each octahedral facet in 2 ways:

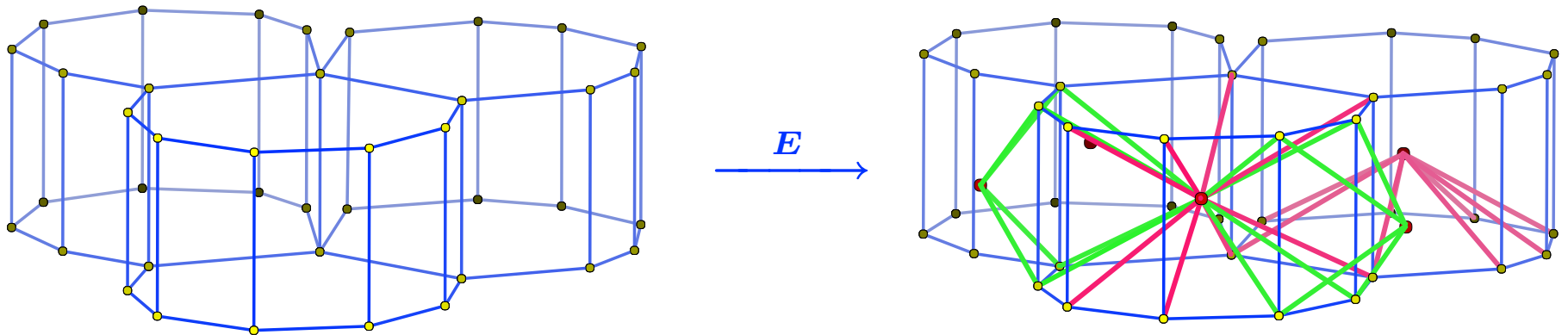
$2^{\Omega(n^c)}$  3-spheres



- ▶ Most of them are distinct: Only  $\leq n!$  combinatorial automorphisms, and

$$n! = 2^{\Theta(n \log n)} \ll 2^{\Omega(n^c)}$$

# The E-Construction [EKZ 2002]

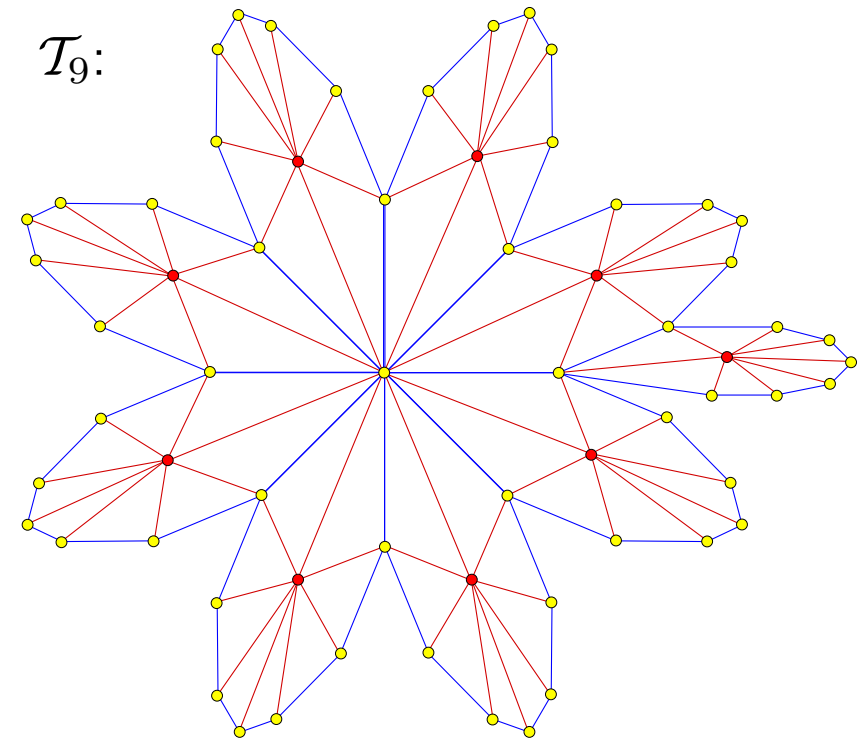
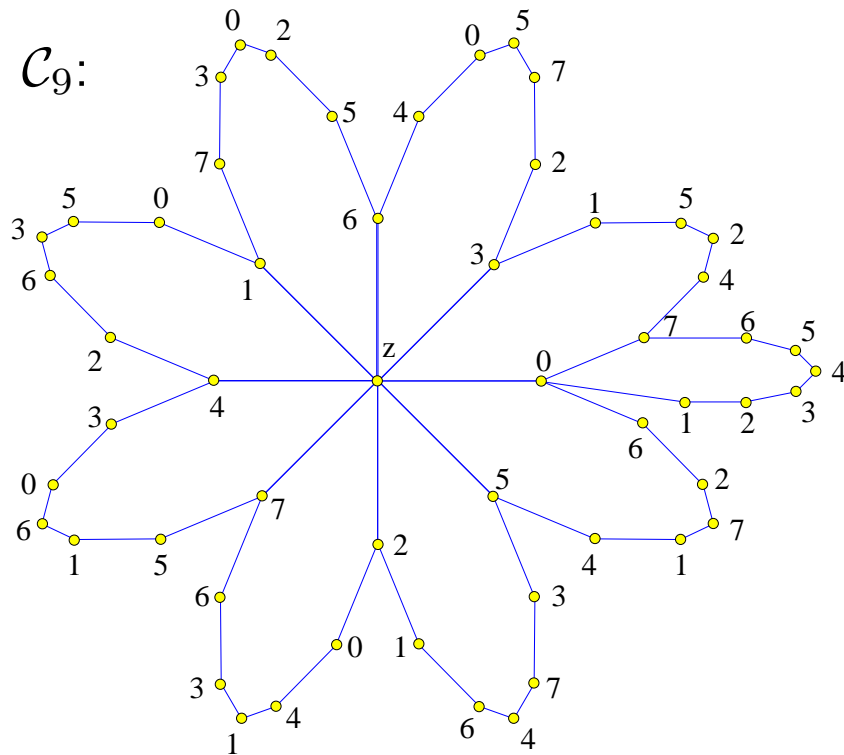


$$\begin{aligned}
 f(\cdot) &= (f_0, f_1, f_2, f_3) & \implies & f(E(\cdot)) = (f_0 + f_3, \cdot, \cdot, f_2) \\
 &= (n, \Theta(n^c), \Theta(n^c), \Theta(n)) & & = (\Theta(n), \cdot, \cdot, \Theta(n^c))
 \end{aligned}$$

**$\Theta(n^c)$  of the 3-cells are octahedra!**

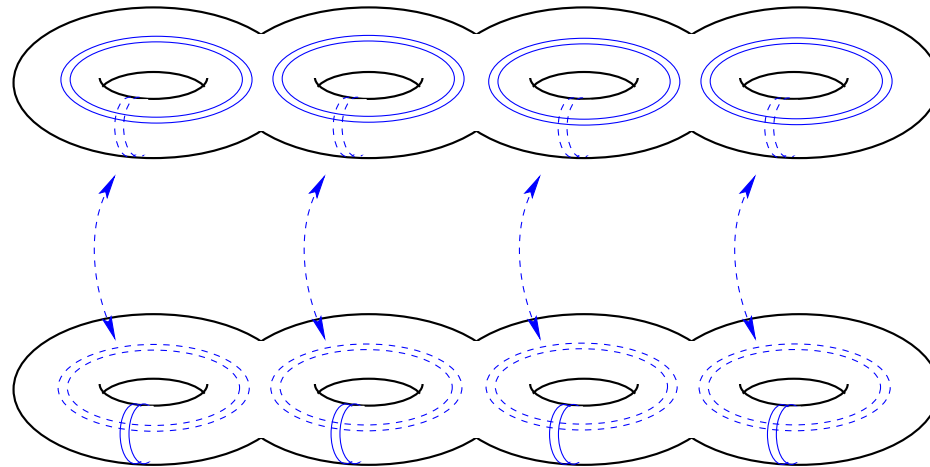
triangulate the rest . . .

# Heffter's [1898] embedding of the complete graph



- ▶ Uses arithmetic in the finite field  $\mathbb{F}_q$  (here:  $q = 9$ )
- ▶ Yields cellulation  $\mathcal{C}_q$  of the surface  $S_g$  of genus  $g = q(q - 5)/4 + 1$
- ▶ This cellulation  $\mathcal{C}_q$  has  $f$ -vector  $(q, \binom{q}{2}, q)$

# Heegaard splittings



**Theorem.** For any  $g \geq 1$ , the 3-sphere may be decomposed into two solid handlebodies  $S^3 = H_1 \cup H_2$  that are identified along a surface  $S_g = H_1 \cap H_2$  of genus  $g$ .



# Many triangulated 3-spheres

- ▶ Start with Heegaard splitting of  $S^3$  of genus  $g$ :

$$S^3 = H_1 \cup H_2, \quad H_1 \cap H_2 = S_g$$

- ▶ Draw Heffter triangulation  $\mathcal{T}_q$  on  $S_g$

- ▶ Thicken boundary:

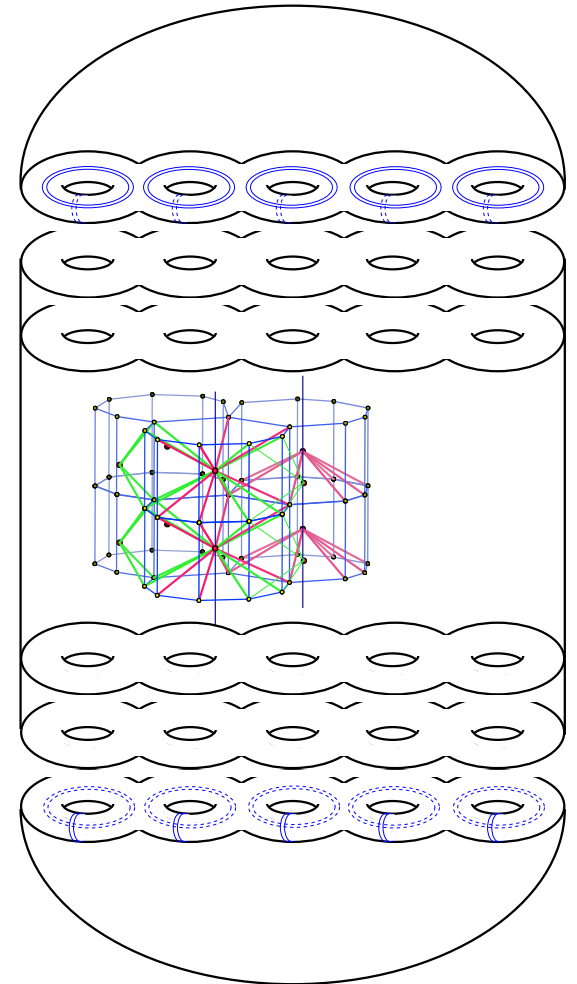
$$S_g \mapsto S_g \times I, \quad \mathcal{T}_q \mapsto \mathcal{T}_q \times I$$

- ▶ **Triangulate  $H_1$  and  $H_2$ :**

$$f(H_i) = (O(q^4), O(q^4), O(q^4), O(q^4))$$

- ▶ Subdivide the interval  $I$  into  $\Theta(q^3)$  pieces and apply E-construction in stack of prisms

$$f(C') = (\Theta(q^4), \Theta(q^5), \Theta(q^5), \Theta(q^5))$$



## Open questions

- ▶ Improve the bounds:  $S(3, n) \gg 2^{\Omega(n^c)}$ ,  $5/4 \leq c \leq 2$
- ▶ Show non-polytopality directly
- ▶ Are these spheres e.g. shellable?