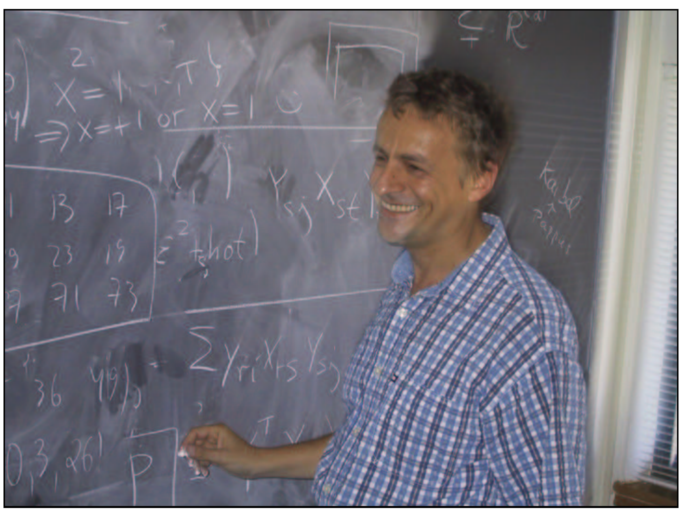
Bernd Sturmfels: Hewlett–Packard Visiting Research Professor, 2003–2004

Professor Bernd Sturmfels of the University of California at Berkeley, and a member at MSRI in the programs on Discrete and Computational Geometry and Topology of Real Algebraic Varieties, is our Hewlett-Packard Visiting Research Professor for the 2003–2004 academic year. Sturmfels has been at UCB since 1995, was a fellow of the Sloan foundation 1991–1993 and a David and Lucile Packard Fellow in the period 1992–1997. He helped organize the MSRI programs in Symbolic Computation in Geometry and Analysis in Fall 1998 and in Commutative Algebra, 2002–2003.

Since 1999, Hewlett-Packard Laboratories have provided funds to support a Hewlett-Packard Visiting Research Professor at MSRI, selected by a joint MSRI/HP Scientific Committee. This position has been filled by researchers at the top of their fields who come to MSRI to do research and give scientific guidance. The Hewlett-Packard Visiting Research Professor also collaborates with researchers at Hewlett-Packard Labs on problems of mutual interest, often involving graduate students and postdoctoral fellows in these projects. The past holders of this position have been Richard Karp, Hendrik W. Lenstra, Gabor T. Herman, Sergio Verdu and Sandu Popescu.

Sturmfels will work with researchers at HP on problems related various areas of interest to the Labs, including coding theory, computational geometry, and algebraic aspects of statistical modeling.



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Motivic Integration

Karen E. Smith

Disclaimer: The author is a believer in the white lie as a way to make technical subjects more accessible. A more accurate survey of this topic is the paper of Looijenga: math.AG/0006220.

Motivic integration was introduced by M. Kontsevich in a 1995 lecture in Orsay, where he announced an affirmative solution to the following conjecture of Batyrev: *two birationally equivalent Calabi–Yau manifolds have the same Hodge numbers.* This conjecture was motivated by work in theoretical physics, namely string theory, which predicts that the universe has a compact direction that is a Calabi–Yau manifold. The conjecture means that two such special manifolds share the same important numerical invariants, provided that they contain large enough isomorphic open subsets.

Kontsevich gave a remarkably elegant and conceptual proof of this result, essentially as a corollary of his theory of motivic integration. He was inspired by the theory of p-adic integration, which Batyrev himself had used to prove a weaker form of his conjecture.

Motivic Integration

There is an associated theory of motivic integration for each smooth complex variety. Like any integration theory, we need

1. a space on which the integration theory is defined,
2. a measure on (certain) subsets of that space,
3. some interesting measurable functions, and
4. an understanding of how integrals transform under change of coordinates.

The measures and functions take values not in the real numbers, but in a more exotic ring $\mathcal{M}$, called the *motivic ring.* The elements of $\mathcal{M}$, roughly speaking, are formal $\mathbb{Z}$-linear combinations of varieties with addition corresponding to disjoint union of varieties and multiplication corresponding to direct products of varieties. This ring is not ordered, so properties of a measure such as "subadditive on unions" do not make sense. However, other basic features of measure theory, such as additivity on finite disjoint unions, do carry over. We discuss each of these four items below.

(1) The arc space of X. Fix a smooth complex variety X. Motivic integration theory for X is an integration theory on the *arc space* $J_\infty(X)$ of X. The points of this space correspond to the "formal arcs" on X. An arc on X can be thought of as an infinitesimal curve centered at a point of X — that is, an arc is a choice of a point, together with a tangent direction at that point, a second order tangent direction, a third order tangent direction and so on. Thinking in this way, it is easy to believe that $J_\infty(X)$ is an infinite dimensional affine bundle over X. Technically speaking, an arc is a map of schemes Spec$\mathbb{C}[[t]] \to X$.

(2) The measure. Motivic integration theory assigns to certain subsets of the arc space, called *cylinder sets*, a value from the motivic ring $\mathcal{M}$. Roughly speaking, a cylinder set is a set in the infinite dimensional space $J_\infty(X)$ defined by a finite amount of data.

Although the arc space $J_\infty(X)$ is infinite dimensional, it is approximated by finite dimensional varieties $J_m(X)$ called the *jet schemes* of X. The space $J_m(X)$ parameterizes the m-*truncated arcs* on X, where an m-truncated arc is can be thought of the choice of a point together with the choice of an i-th order tangent direction at that point for all $i \leq m$. There is a natural *truncation map*

$$\pi_m : J_\infty(X) \to J_m(X)$$

sending each arc to its truncation at the m-th term.

A cylinder set is a subset of the arc space $J_\infty(X)$ obtained from one of these finite dimensional approximating spaces $J_m(X)$. More precisely, a cylinder set is any subset of $J_\infty(X)$ of the form $\pi_m^{-1}(B_m)$, where B_m is a subvariety of $J_m(X)$. In this case, we can assign to the set $B = \pi_m^{-1}(B_m)$ the measure

$$\mu_X(B) = [B_m] \cdot \mathbb{L}^{-nm},$$

where $[B_m]$ is the class of B_m in $\mathcal{M}$, $\mathbb{L}$ is the class of the affine line in $\mathcal{M}$ (so that $\mathbb{L}^{-1}$ is its formal multipicative inverse) and n is the dimension of X. It is easy to check that this measure is well-defined, using the fact that the natural truncation maps from $J_{m+1}(X)$ to $J_m(X)$ are affine bundles of fiber dimension n.

As a quick example, which is incidentally crucial to the proof of Kontsevich's theorem, let us compute the integral of the constant function 1 over the whole arc space $J_\infty(X)$. Note that $J_\infty(X)$ is the preimage of the "total" truncation map $J_\infty(X) \to J_0(X) = X$. Thus its measure is

$$\mu(J_\infty(X)) = [X] \cdot \mathbb{L}^{n \cdot 0} = [X].$$

Therefore

$$\int_{J_\infty(X)} d\mu_X = [X], \quad \text{the class of } X \text{ in } \mathcal{M}.$$

(3) Some interesting functions to integrate. Now fix any subvariety D of our smooth variety X. The subvariety D gives rise to an integer valued function F_D on the arc space $J_\infty(X)$, given, roughly speaking, by order of tangency of each arc along D. The value of $F_D(\gamma)$ may be zero (when γ misses D completely) or infinite (when γ is tangent to D to every order), or any finite number in between. The definition extends naturally to the case where D is a *divisor* on X, meaning a formal Z-combination of codimension one subvarieties of X. In technical terms, the value of F_D along an arc

$$\gamma^* : \operatorname{Spec} C[[t]] \to X$$

is the degree of the pullback divisor $\gamma(D)$ on the curve $\operatorname{Spec} C[[t]]$.

One example of an interesting measurable function on $J_\infty(X)$ is the function $\mathbb{L}^{-F_D}$. Because it takes discrete values, its integral over $J_\infty(X)$ is really just a large sum, which can be shown to converge in $\mathcal{M}$ (in a suitable topology).

(4) Kontsevich's Birational Transformation Rule. The result that gives the theory of motivic integration its power is a formula for how motivic integrals transform under birational change of space. Specifically, let $f : X \to Y$ be a proper birational morphism of smooth algebraic varieties. The Jacobian of this map defines a divisor $K_{X/Y}$ called the *relative canonical divisor* of the map. Algebraic geometers also recognize $K_{X/Y}$ as the unique divisor supported entirely on the exceptional set of f and belonging to the divisor class $K_X - f^*(K_Y)$, where K_X and K_Y denote the divisor classes associated to the canonical bundles on X and Y respectively. In its simplest form, Kontsevich's birational transformation rule states that for any divisor D on Y,

$$\int_{J_\infty(Y)} \mathbb{L}^{-F_D}\, d\mu_Y = \int_{J_\infty(X)} \mathbb{L}^{-F_{f^*(D)+K_{X/Y}}}\, d\mu_X.$$

This theorem is non-trivial to prove, but essentially uses only fairly basic commutative algebra.

The Proof of Kontsevich's Theorem

Having outlined this much of the theory of motivic integration, it is easy to understand Kontsevich's proof of the opening conjecture. First let us first understand the conjecture more precisely.

A Calabi–Yau manifold is a smooth complex projective algebraic variety of dimension n admitting a nowhere vanishing holomorphic n-form — in other words, it has trivial canonical bundle. (The precise definition of a Calabi–Yau manifold also requires certain cohomology groups to vanish, but this assumption is not necessary for Kontsevich's theorem.) Two complex manifolds are birationally equivalent if there are mutually inverse maps between them given locally by rational functions in local coordinates; these maps need not be defined on the closed sets where the denominators of the rational functions vanish. For any non-negative integers p and q, the Hodge number $h_{pq}(X)$ is the dimension of the space of closed (smooth) pq-forms on X modulo the space of exact pq-forms on X; alternatively, $h_{pq}(X)$ is the dimension of the cohomology group $H^p(X, \Omega_X^q)$ where Ω_X^q is the sheaf of holomorphic q-forms on X. Putting this all together, Kontsevich's theorem states:

Theorem. *If X and Y are birationally equivalent smooth complex projective varieties, each of whose canonical bundle is trivial, then $h_{pq}(X) = h_{pq}(Y)$ for all $p, q \geq 0$.*

To prove this, Kontsevich considers a map from the set of all smooth projective varieties to the polynomial ring $Z[u, v]$, defined by sending a variety X to the polynomial

$$\sum_{pq} h_{pq}(X) u^p v^q.$$

His idea is that (using some basic Hodge theory) this map factors through the motivic ring $\mathcal{M}$. The point is then to show that two birationally equivalent Calabi–Yau manifolds map to the same element in $\mathcal{M}$. Therefore, mapping further to $Z[u, v]$, they must of course have the same Hodge numbers.

In factoring the map through $\mathcal{M}$, a variety X is sent to the element it determines in $\mathcal{M}$, or its class $[X]$. As we computed above, the class $[X]$ can be interpreted as the integral

$$\int_{J_\infty(X)} d\mu_X = [X].$$

Noting that the constant function 1 can be written as $\mathbb{L}^{-F_\emptyset}$, where $\emptyset$ is the trivial divisor, we also have

$$\int_{J_\infty(X)} \mathbb{L}^{-F_\emptyset}\, d\mu_X = [X].$$

Now the point is to use the birational transformation rule to compute this integral in a different way. Let Z be any smooth variety admitting a proper birational morphism $g : Z \to X$ to a variety X. If X has trivial canonical bundle, then

$$K_{Z/X} = K_Z - g^*(K_X) = K_Z,$$

which is independent of X. So using Kontsevich's birational transformation rule, we see that

$$[X] = \int_{J_\infty(X)} \mathbb{L}^{-F_\emptyset}\, d\mu_X = \int_{J_\infty(Z)} \mathbb{L}^{-F_{g^*(\emptyset)+K_{Z/X}}}\, d\mu_Z$$
$$= \int_{J_\infty(Z)} \mathbb{L}^{-F_{K_Z}}\, d\mu_Z.$$

The remarkable thing to notice about this computation is that $[X]$ depends only on Z. But, if X and Y are birationally equivalent, there always exists a smooth variety Z admitting a proper birational morphism to both X and Y (a so-called *resolution of indeterminacies*). So in this case, if both X and Y have trivial canonical bundles, then the classes $[X]$ and $[Y]$ in $\mathcal{M}$ can both be computed as the same motivic integral on Z. This means that X and Y have the same image in $\mathcal{M}$, and hence the same image in $Z[u, v]$ and the same Hodge numbers.

Other Applications of Motivic Integration

This beautiful proof of Kontsevich is only the beginning of a long story featuring many elegant applications of motivic integration to problems in algebraic and arithmetic geometry. For one thing, Denef and Loeser have developed the theory for singular varieties X, as many applications require. In our seminar, we focussed on applications to higher dimensional birational geometry, a program that started with the 2001 Berkeley PhD thesis of Mircea Mustata. For example, using motivic integration, Mustata shows that a singular hypersurface X has *rational singularities* if and only if the jet schemes $J_m(X)$ are all irreducible. Furthermore, he also finds formulas for a subtle invariant called the *log-canonical threshold* of a hypersurface singularity in terms of the dimensions of these jet schemes. Later work of Ein, Mustata and Yasuda establishes a version of the famous *inversion of adjunction* conjecture, which is then applied to prove the following long open conjecture: *no smooth hypersurface of degree n in P^{n+1} is birationally equivalent to projective space, for* $4 \le n \le 12$.

The elegant proofs of these concrete results are just some of the topics we studied in our Motivic Integration seminar in 2002–2003 at MSRI, as part of the Special Year in Commutative Algebra. Some of our notes and a collection of electronic literature can be found on our seminar web site: www.mabli.org/jet.html.

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MSRI Goes Wireless

To cater to the needs of the more than 1500 scientists who, every year, spend anywhere from a week to a many months at MSRI, we installed as far back as late 2002 a simple wireless network so that users, using their laptops, may connect to the network from most parts of the building. This is especially important for short-term visitors who don't have an office.

The current configuration uses both a Linksys 2 Access Port and Apple Airports (the latter donated by Chris Heegard). To have wireless access, incoming members and workshop participants need to have a wireless card in their computer (a few cards are available for borrowing), and must also receive an authentication key from computing personnel.

Because some areas, like the lecture hall, don't receive signals well because of interference or blockage, Rachelle Summers (Head of Computing) and Max Bernstein (Network Administrator) are currently improving the wireless system. The upgrade, scheduled to roll out before the end of 2003, will likely involve additional access ports and the installation of antennas, and possibly a switch from the current 802.11b protocol, which tranfers 11 megabits per second, to 802.11g, which tranfers up to 54 Mbits/s.

MSRI seeks to offer its members and visitors the best possible facilities. Fortunately even a very reliable and capacious system, capable of supporting wireless use even outdoors around the building, is quite affordable nowadays.

Rachelle Summers, MSRI Head of Computing