Complex Hyperplane Arrangements

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We were fortunate to spend the 2004 fall semester in residence at MSRI, participating in the program on Hyperplane Arrangements and Applications. It was an intense, stimulating, productive, enlightening, eventful and most enjoyable experience. It was especially so for us long-timers in the field because the program truly marked a coming-of-age in the evolution of the subject from relative obscurity thirty years ago. We had an opportunity to introduce a group of graduate students to the wonders of arrangements during the two-week MSRI graduate school in Eugene in early August, and an impressive group of post-docs, along with many other unsuspecting mathematicians, during the program. We are glad to have this chance to bring some of the ideas to a wider audience. For further reference, we suggest the reader consult the books and survey articles listed on the summer school web page, www.math.neu.edu/~suciu/eugene04.html, and the references therein.

In its simplest manifestation, an arrangement is merely a finite collection of lines in the real plane. The complement of the lines consists of a finite number of polygonal regions. Determining the number of regions turns out to be a purely combinatorial problem: one can easily find a recursion for the number of regions, whose solution is given by a formula involving only the number of lines and the numbers of lines through each intersection point. This formula generalizes to collections of hyperplanes in \( \mathbb{R}^t \), where the recursive formula is satisfied by an evaluation of the characteristic polynomial of the (reverse-ordered) poset of intersections. The study of characteristic polynomials forms the backbone of the combinatorial, and much of the algebraic theory of arrangements, which were featured in the MSRI Workshop on Combinatorial Aspects of Hyperplane Arrangements last November.

From the topological standpoint, a richer situation is presented by arrangements of complex hyperplanes, that is, finite collections of hyperplanes in \( \mathbb{C}^t \) (or in projective space \( \mathbb{P}^t \)). In this case, the complement is connected, and its topology, as reflected in the fundamental group or the cohomology ring for instance, is much more interesting.

The motivation and many of the applications of the topological theory arose initially from the connection with braids. Let \( A_\ell \equiv \{ z_i = z_j \mid 1 \leq i < j \leq \ell \} \) be the arrangement of diagonal hyperplanes in \( \mathbb{C}^\ell \), with complement the configuration space \( X_\ell \). In 1962, Fox and Neuwirth showed that \( \pi_1(X_\ell) \cong \mathbb{F}_\ell \), the pure braid group on \( \ell \) strings, while Neuwirth and Fadell showed that \( X_\ell \) is aspherical. A few years later, as part of his approach to Hilbert’s thirteenth problem, Arnol’d computed the cohomology ring \( H^+(X_\ell, \mathbb{C}) \).

For an arbitrary hyperplane arrangement in \( \mathbb{C}^t \), the fundamental group of the complement, \( G = \pi_1(X) \), can be computed algorithmically, using the braid monodromy associated to a generic projection of a generic slice in \( \mathbb{C}^2 \). The end result is a finite presentation with generators \( x_i \) corresponding to meridians around the \( n \) hyperplanes, and commutator relators of the form \( x_i x_j x_i^{-1} x_j^{-1} \), where \( x_j \in P_n \) are the (pure) braid monodromy generators, acting on the meridians via the Artin’s representation \( P_n \rightarrow \text{Aut}(F_n) \).

The cohomology ring \( H^+(X, \mathbb{Q}) \) was computed by Brieskorn in the early 1970’s. His proof shows that \( X \) is a formal space, in the sense of Sullivan: the rational homotopy type of \( X \) is determined by \( H^+(X, \mathbb{Q}) \). In particular, all rational Massey products vanish. In 1980, Orlik and Solomon gave a simple combinatorial description of the \( k \)-algebra \( H^+(X, k) \), for any field \( k \): it is the quotient \( A = E/I \) of the exterior algebra \( E \) on classes dual to the meridians, modulo a certain ideal \( I \) determined by the intersection poset.

For each \( a \in A^1 \cong k^n \), the Orlik–Solomon algebra can be turned into a cochain complex \( (A, a) \), with \( i \)-th term the degree \( i \) graded piece of \( A \), and with differential given by multiplication by \( a \). The *resonance varieties* of \( A \) are the jumping loci for the cohomology of this cochain complex:

\[
R_d(A) = \{ a \in A^1 \mid \dim_k H^i(A, a) \geq d \}.
\]

The case of a line arrangement in \( \mathbb{P}^2 \) is already quite fascinating. The subarrangements that contribute components to \( R_d(A) \) have very special combinatorial and geometric properties. To be eligible, the incidence matrix for the lines and intersection points must have null-space of dimension at least two, with full support. In addition, the subarrangement must have a partition into at least three classes such that no point \( p \) is incident with one line of one class, while all other lines incident with \( p \) belong to a second class. Such partitions are termed *neighborly*. The simplest nontrivial example is provided by the braid arrangement \( A_3 \):

In this figure the points represent hyperplanes and the lines correspond to the points of multiplicity greater than two. This is a diagram of the matroid associated with the arrangement. When \( k \) has characteristic zero, the Vinberg classification of generalized Cartan matrices implies an even more exceptional situation. One can assign multiplicities to the lines so that the partition is into three classes such that no point \( p \) is incident with one line of one class, while all other lines incident with \( p \) belong to a second class. Such partitions are termed neighborly. The simplest nontrivial example is provided by the braid arrangement \( A_3 \):

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The singular fibers are given by $a = 0$, $b = 0$, and $c = 0$. A nonreduced example is provided by the arrangement of symmetry planes of the cube with vertices $\{\pm 1, \pm 1, \pm 1\}$, with the coordinate hyperplanes having multiplicity two. This multi-arrangement is interpolated by a pencil of quartics. Such pencils often yield (nonlinear) fiberings of the complement by punctured surfaces, showing in particular that the complement is aspherical.

There is also an apparent connection between the cohomology of $A(X, \alpha)$ and critical points of certain multivariate rational functions. A resonant degree-one element $\alpha$ is represented (up to a scalar) by a logarithmic deRham one-form $d \log \Phi$, where $\Phi$ is a product of the defining linear forms of the hyperplanes, raised to integral powers. The dimension of $H^i(A, \alpha)$ is related to the number of components in the critical locus of $\Phi$ of codimension $i$. In particular we expect $\Phi$ to have nonisolated critical points when $\alpha$ is "generically resonant." This is known to be the case for certain high-dimensional arrangements with certain weights, and was established for arrangements of rank three during the Fall program. A precise description of this relationship in general is a topic of current study.

In our example of the $A_3$ arrangement, $d \log \Phi$ is resonant precisely when

$$\Phi(x, y, z) = (x^2 - y^2)\alpha(y^2 - z^2)\beta(z^2 - x^2)\gamma,$$

with $\alpha + \beta + \gamma = 0$. The critical set $d \Phi = 0$ is given (projectively) by

$$[x^2 - y^2 : y^2 - z^2 : z^2 - x^2] = [\alpha : \beta : \gamma].$$

It is not a coincidence that these critical loci are curves in the pencil shown in the previous figure.

Through the connection with generalized hypergeometric functions, the critical locus of $\Phi$ is of interest in relation to the Bethe Ansatz in mathematical physics. This was a major topic of discussion in the MSRI workshop on Topology of Arrangements and Applications last October. Somewhat serendipitously, the same answer to the linearity question is a singular cubic threefold in $\mathbb{P}^4$ ruled by lines, in characteristic three. The underlying arrangement is the Hessian arrangement of 12 lines determined by the intersection points on a general cubic.

As noted by Rybnikov, the fundamental group of the complement, $G = \pi_1(X)$, is not necessarily determined by the intersection poset. Even so, the ranks $\phi_k(G)$ of the successive quotients of the lower central series $\{G_k\}_{k \geq 1}$, where $G_1 = G$ and $G_{k+1} = [G_k, G_k]$, are combinatorially determined. Indeed, by a classical result of Sullivan, the formality of $X$ implies that the graded Lie algebra $gr(G) = \bigoplus_{k \geq 1} G_k/G_{k+1}$ is rationally isomorphic to the holonomy Lie algebra $h_A$ associated to $A = H^*(X; \mathbb{Q})$. Furthermore, it was recently shown that the Chen Lie algebra, $gr(G/G''')$, is rationally isomorphic to $h_A/h_A''$, and so the Chen ranks $\theta_k(G)$ are also combinatorially determined.

Much effort has been put in computing explicitly the LCS and Chen ranks of an arrangement group $G$. It turns out that both can be expressed in terms of the Betti numbers of the linear strands in certain free resolutions (over $A$ or $E$):

$$\prod_{k=1}^{\infty} (1 - t^k) \phi_k(G) = \sum_{i=0}^{\infty} \dim \text{Tor}_i^A(Q, Q) t^i,$$

$$\theta_k(G) = \dim \text{Tor}_{k-1}^E(A, Q)_k, \quad \text{for } k \geq 2.$$

When the arrangement is of fiber-type (equivalently, the intersection lattice is supersolvable), $A$ is a Koszul algebra. The first formula immediately above, together with Koszul duality, yields the classical LCS formula of Kohno and Falk–Randell:

$$\prod_{k=1}^{\infty} (1 - t^k) \phi_k(G) = \text{Hilb}(A, -t).$$

In a survey paper from 2001, one of us (Falk) made two conjectures, expressing (under some conditions) the LCS and Chen ranks of an arrangement group in terms of the dimensions of the characteristic varieties, yields information about the homology of finite abelian covers of the complement. This approach gives a practical algorithm for computing the Betti numbers of the Milnor fiber $F$ of a central arrangement in $\mathbb{C}^3$. It has also led to examples of multi-arrangements with torsion in $H_1(F)$. There are no known examples of ordinary arrangements with this property.
the components of the first resonance variety. Write $R_1^d(A) = L_1 \cup \cdots \cup L_q$, with $\dim L_i = d_i$. Then, conjecturally,

$$\prod_{k=2}^{\infty} (1 - t^k) \phi_k(G) = \prod_{i=1}^q \frac{1 - t d_i t}{(1 - t)^{d_i}}$$

provided $\phi_4(G) = \theta_4(G)$, and

$$\theta_k(G) = (k - 1) \sum_{i=1}^q \binom{k + d_i - 2}{k}$$

for $k$ sufficiently large.

The inequality $\geq$ implicit in this second formula has been proved by Schenck and Falk. The reverse inequality has a compelling algebro-geometric interpretation in terms of the sheaf on $\mathbb{C}^n$ determined by the linearized Alexander invariant. Equality in both of the formulas just given has been verified in a number of papers for two important classes of arrangements: decomposable arrangements (essentially, those for which all components of $R_1^d(A)$ arise from sub-arrangements of rank two), and graphic arrangements (i.e., sub-arrangements of the braid arrangement).

Many of the results and observations reported on here represent joint work (or work in progress) with our friends and collaborators: Dan Cohen, Graham Denham, Dani Matei, Stefan Papadima, Hal Schenck, Sasha Varchenko, and Sergey Yuzvinsky. Our thanks go to them. In addition, we are grateful to many other unnamed participants in the MSRI program last Fall, for the countless hours spent in helpful and stimulating conversations about arrangements.

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