THE EIGHTFOLD WAY: **ART FROM THE KLEIN QUARTIC**

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The German mathematician Felix Klein discovered in 1878 that a certain surface, now known as the Klein quartic, has a number of remarkable properties, including an incredible 336-fold symmetry, the maximum possible degree of symmetry for any surface of its type. Since then, mathematicians have discovered that the same object comes up in different guises in many areas of mathematics, from complex analysis and geometry to number theory.

The surface cannot be represented in our everyday three-space in such a way that all the symmetries are obvious, but one way to visualize them is this. The surface is covered by 24 heptagons, forming a completely regular pattern — just as regular as the pattern of pentagons on a dodecahedron. Place a finger on any edge of any heptagon. Trace out along the edge to the next intersection, and turn left. Now go to the next intersection and turn right. Continue in this way, making a total of 8 turns, LRLRLRLR, and you will arrive back where you started. It doesn't matter where you start or in which direction you go: in eight alternating turns, you always arrive back at the beginning.

This explains the name "The Eightfold Way" given by Helaman Ferguson to his sculpture representing the Klein quartic; the 24 heptagons are shown on the surface of its Carrara marble, and one particular eightfold way is shown by the raised edges, while the remaining edges are grooves.

The base of the sculpture, in Virginia black serpentine, shows a tiling in hyperbolic, or non-Euclidean, geometry. The tiling of the hyperbolic plane is an unwrapping of the surface, just as a tiling of the plane by squares can be thought of as the unwrapping of a torus. If we think of the hyperbolic plane as the complex half-plane (the set of complex numbers with positive imaginary coordinates), this tiling is related to the modular group $\Gamma(7)$ of 2×2 matrices with integer coordinates and congruent to the identity modulo 7. It was while investigating this group of transformations of the upper half-plane that Klein discovered his marvelous surface. He then showed that the resulting Riemann surface could be represented by the equation $x^3y + y^3z + z^3x = 0$ in complex projective coordinates, and that any other representation as a plane quartic is equivalent by a change of variables.

This simple quartic equation is at the root of many connections with other fields of mathematics. For example, in number theory, the Klein equation is related to the equation $a^7 + b^7 = c^7$, a particular case of Fermat's equation.

For more information on the Klein quartic, see *The Eightfold Way*, edited by Silvio Levy (MSRI Publications vol. 35, Cambridge University Press, 1999). For more information on the artist and his work, see *Helaman Ferguson: Mathematics in Stone and Bronze*, by Claire Ferguson (Meridian Creative Group, 1994). Both books are available in the MSRI library.