Some Open and Relevant Mathematical Questions Related to Nonextensive Statistical Mechanics

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1.

One-dimensional unimodal dissipative maps (e.g., the logistic map) are known today to be deeply related to nonextensive statistical mechanics. Their sensitivity to the initial conditions satisfies

$$\xi = e_q^{\lambda_q t},$$

where

$$e_q^x = [1 + (1-q)x]^{1/(1-q)} \qquad (e_1^x = e^x).$$

For all values of the control parameter of the map for which the Lyapunov exponent λ_1 is different from zero, we have $q = 1$, hence $xi = e^{\lambda_1 t}$, as is well known.

But for values of the control parameter for which the Lyapunov exponent λ_1 vanishes (e.g., edge of chaos, doubling-period bifurcations, tangent bifurcations or intermittency), we have a power-law sensitivity associated with $q = q_{\rm sen}$, namely

$$\xi = e_{q_{\rm sen}}^{\lambda_{q_{\rm sen}} t},$$

where q_sen is greater than 1 for the period-doubling and the tangent bifurcations, and q_sen is less than 1 for the (very important!) edge of chaos. For example, for the standard logistic map edge of chaos, $q_{\rm sen} = 0.2445\ldots$

All these features have been intensively checked by many authors around the world in a great variety of maps. Such behavior was conjectured in 1997 [C. Tsallis, A.R. Plastino and W.-M. Zheng, "Power-law sensitivity to initial conditions - New entropic representation", *Chaos, Solitons and Fractals* **8**, 885 (1997)]. It has been checked either numerically or "proved" in the style of theoretical physics [in a series of papers by F. Baldovin and A. Robledo]. However, the mathematical rigorous treatment of such questions is still lacking. It would be extremely timely and useful, since we do not yet know under WHAT precise conditions such properties are to be expected. If we knew, that would certainly be useful for identifying physical systems to which such notions should be applicable.

2.

For the same class of maps discussed above, it has been "proved" (in the style of theoretical physics) that, at the edge of chaos,

$$\frac{1}{1-q_{\rm sen}} = \frac{1}{\alpha_{\rm min}} - \frac{1}{\alpha_{\rm max}},$$

where $\alpha_{\rm min}$ and $\alpha_{\rm max}$ are the two values of the Hölder exponent α for which the multifractal function $f(\alpha)$ possibly vanishes. As before, we do not know for WHAT dynamical systems such relation is valid. How this formula becomes generalized if $f(\alpha_{\rm min})$ and $f(\alpha_{\rm max})$ do NOT vanish? Is it true for dimensions above unity?

3.

For the same class of maps discussed above we have that, if $\lambda_1 > 0$, then, in the limit as $t \to \infty$,

$$\frac{S_1}{t} = K_1 > 0,$$

where $S_1 = -sum_i p_i ln p_i$ and K_1 is essentially the Kolmogorov-Sinai entropy rate. Moreover, $\lambda_1 = K_1$ (Pesin theorem). All this has long since been proved by mathematicians.

But what happens when $\lambda_1 = 0$, besides the trivial fact that $K_1 = 0$? At the edge of chaos, we have, as $t \to infty$,

$$S_{q_s en}/t = K_{q_s en} > 0,$$

where

$$S_q = \frac{1}{q-1}\left(1 - sum_i p_i^q\right)$$

Moreover, the q-generalization of Pesin theorem has been conjectured and then "proved", namely that

$$K_{q_{\rm sen}} = \lambda_{q_{\rm sen}}.$$

It would be very useful to have all these facts rigorously proved by mathematicians.

Many more related properties, for both dissipative and conservative maps (e.g., the standard map), are known today by specialists in nonextensive statistical mechanics. All this would certainly benefit at this point from mathematical approaches.

4.

Classical long-range-interacting N-body Hamiltonians are those whose particles interact through a (attractive) potential which decays at long distance r as

$1/r^\alpha$ with $0 \leq \alpha/d < 1$, where d is the dimension of the space (e.g., Newtonian gravitation corresponds to $\alpha = 1$ and $d = 3$). There is strong numerical (and experimental) evidence that such important Hamiltonians exhibit anomalous behavior for a nonzero-measure class of initial conditions. They live in a longstanding metastable (or quasi-stationary) state, and only after this they make a crossover to the usual Boltzmann-Gibbs thermal equilibrium state. The following properties are observed:

1. The duration of the metastable state diverges with N, being therefore the only interesting state for thermodynamically large systems.

2. The maximum Lyapunov exponent vanishes in the limit $N \to \infty$.

3. The distribution of velocities is not Maxwellian, hence the distribution of energies is not that Boltzmann-Gibbs.

4. There is aging in various basic two-body correlation functions.

5. Diffusion is anomalous in phase space.

Based on this and other evidence, a conjecture has been made (by Tsallis) that the system evolves in the full phase space, not in a mixing-ergodic manner, but visiting a scale-free network of the type frequently observed in natural and artificial systems and also discussed by Barabasi and others. The specific network would depend on the particular initial condition but its statistical structure would be the same within that nonzero-measure class of initial conditions. One of the landmarks of this network would be the probability for a site to have k bonds (linking it to other sites) being proportional to $e_q^{-k/\kappa}$, where q is expected to be a monotonically decreasing function $q(\alpha/d)$ such that $q(1) = 1$, remaining unity for all $\alpha/d > 1$.

THE PRESENT LIST IS BY NO MEANS EXHAUSTIVE.