Julia Robinson Mathematics Festival

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Building Blocks

You have an unlimited supply of cubes. Each cube's side length is a number of the form 2^i, where i is any nonnegative integer. That is, there are cubes with edges of length 1 unit, 2 units, 4 units, 8 units, 16 units, and so on.

You wish to build a tower of a given exact height by stacking cubes one on top of another.

Part 1: Building Three Towers
You need to build three towers, of heights 10, 15, and 32.

1. What is the biggest number of cubes you could use to build the three towers?
2. What is the smallest number of cubes you could use to build the three towers?
3. What is the biggest possible total volume of the three towers?
4. What is the smallest possible total volume?
5. What is the largest possible total surface area for the three towers?
6. What is the smallest possible total surface area?

Part 2: Limited Supplies for One Tower
Now you have only m cubes, of edge length N_1, N_2, N_3, …, N_m. For simplicity we'll suppose they go in increasing order, so N_1 is the smallest and N_m is the biggest. There may be ties (or they may even all be the same size). You use them all to build a single tower.

7. What is the height of this tower?
8. What is the total volume of this tower?
9. What is the maximum possible total surface area for this tower?
10. What is the minimum possible total surface area?

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Part 3: Friendly Fun
Your friend builds a tower using at most n blocks of each side length. For instance, if n is 3 then your friend selected blocks from a pile with edges of length 1, 1, 1, 2, 2, 2, 4, 4, 4, 8, 8, 8, and so on. Also, the largest block in your friend's tower has length c, which equals 2^k for some value of k. (You might enjoy visiting the "To Twos" table if you like this part.)

11. What is the largest possible number of cubes that could be in your friend's tower?
12. What is the smallest possible number of cubes in your friend's tower?
13. What is the largest possible height for your friend's tower?
14. What is the smallest possible height?

Part 4: Covering the Field
Now you have a rectangular field, on the ground, that needs to be covered by cubes. No more towers, just every spot covered by a cube, and no cubes sticking out past the edges.

15. The field is 9 units long and 8 units wide. What is the largest number of cubes you can use to cover it?
16. What is the smallest number of cubes you can use to cover it?
17. Now you're allowed to use only one cube of each size. What rectangular fields can you cover?
18. New rules let you use up to two cubes of each size. What rectangular fields can you cover?
19. The field is L units long and W units wide. What is the smallest number of cubes needed to cover the field? Hint: start with small examples and work your way up in an organized table.
20. Write a formula expressing your solution method in the previous problem. You may find it useful to use the function $F(x, y)$ = the largest power of 2 that is less than or equal to both x and y in expressing your answer. Also, you may find it useful in writing your formula to assume that you have already figured out the fewest possible cubes for all smaller fields; feel free to use that function in your answer as well.

Part 5: The Third Dimension

Now we're going to fill solid spaces with cubes.

21. What is the fewest cubes it takes to fill a 7 by 6 by 5 box?

22. What is the fewest cubes needed to fill an x by y by z box? Do the same approaches you used in the previous problem still work?

23. Can you write a general formula?

24. What other questions could you ask here? For example, what kinds of limitations of the number of cubes of each size would be interesting? What happens if you only have one cube of each size?

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