TAKING IT FURTHER

[The following passage is adapted from the entry "Gaussian elimination" that appears in *The Facts On File Encyclopedia of Mathematics* by James Tanton.]

Let's consider another system of equations and apply a more systematic approach to finding a solution to it. Consider, for example, the three equations in three unknowns:

$$\begin{aligned} y + 3z &= 0 \\ 2x + 4y - 2z &= 18 \\ x + 5y + 3z &= 14 \end{aligned}$$

The method we shall follow is called "Gaussian elimination" named in honor of the great German mathematician and physicist Carl Friedrich Gauss (1777-1855). The idea is to multiply selected equations with carefully chosen constants and subtract equations to eliminate variables. (This is also called "pivoting.") Here goes:

Let's begin by interchanging the first two equations so that the variable *x* appears in the first row:

$$\begin{aligned} 2x + 4y - 2z &= 18 \\ y + 3z &= 0 \\ x + 5y + 3z &= 14 \end{aligned}$$

Now divide the first equation through by 2 so that the coefficient of the leading variable *x* is one.

$$\begin{aligned} x + 2y - z &= 9 \\ y + 3z &= 0 \\ x + 5y + 3z &= 14 \end{aligned}$$

Eliminate the appearance of the variable *x* from the third equation by subtracting the first equation from it. That is, replace the third equation with the equivalent statement:

$(x+5y+3z)-(x+2y-z)=14-9$, that is, $3y+4z=5$. We have:

$$\begin{aligned} x+\ 2y-\ \ z &= 9 \\ y+3z &= 0 \\ 3y+4z &= 5 \end{aligned}$$

The second equation contains *y* as the leading variable with a coefficient of one. Eliminate *y* from the third equation by subtracting from it three copies of the second equation, that is, replace third equation with the equivalent statement, $(3y+4z)-3(y+3z)=5-3\cdot 0$, that is, $-5z=5$. We now have:

$$\begin{aligned} x+\ 2y-\ \ z &= 9 \\ y+3z &= 0 \\ -5z &= 5 \end{aligned}$$

Divide the third equation through by –5 so that the leading variable in it is *z* with a coefficient of one:

$$\begin{aligned} x+\ 2y-\ \ z &= 9 \\ y+3z &= 0 \\ z &= -1 \end{aligned}$$

The solution to the system of equations is now easy to compute. By a process of “back substitution” we see that $z=-1$, from which it follows from the second equation that $y=-3z=3$, and from the first equation that $x=9-2y+z=9-6-1=2$. One checks that this is indeed the solution to the original set of equations.

SOME TERMINOLOGY:

If you have taken a course in linear algebra you may have heard the following terms:

An "elementary row operation" is any maneuver on a set of linear equations (that is, equations that involve no products of variables or variables raised to powers other than one) that:

i) interchanges two equations

ii) multiplies (or divides) an equation through by a non-zero quantity

iii) adds or subtracts a multiple of one equation from another.

The process of Gaussian elimination uses elementary row operations to transform a system of linear equations into an equivalent system in "echelon form," that is, one in which each equation leads, in turn, with one of the variables with coefficient one. Via the process of "back substitution" it is straightforward to then determine the solution to the system of equations. The process illustrated above works for any number of equations in the same number of unknowns, and, as we shall see below, can easily be adapted for systems with more variables than equations. It is possible that during the process of Gaussian elimination a system of equations might yield an absurd statement (such as $0 = 9$, for instance) in which case one would conclude that the system has no solutions, or possibly a vacuous statement (such as, $0 = 0$), in which case one would conclude that the system of equations has infinitely many solutions. Here are some examples of the possible scenarios that can arise:

A SYSTEM WITH NO SOLUTION:

Consider the following set of equations in three unknowns:

$$\begin{aligned} x + 3y + 3z &= 15 \\ x + 2y + z &= 14 \\ 2y + 4z &= 11 \end{aligned}$$

Subtract the first equation from the second to remove the appearance of variable *x* from the second line. This gives:

$$\begin{aligned} x + 3y + 3z &= 15 \\ -y - 2z &= -1 \\ 2y + 4z &= 11 \end{aligned}$$

Now multiply the second equation by negative one:

$$\begin{aligned} x + 3y + 3z &= 15 \\ y + 2z &= 1 \\ 2y + 4z &= 11 \end{aligned}$$

To remove the appearance of the variable *y* from the third line, subtract double the second equation from the third:

$$\begin{aligned} x + 3y + 3z &= 15 \\ y + 2z &= 1 \\ 0 &= 9 \end{aligned}$$

Note that final statement is absurd! This means that there can be no numbers *x*, *y*, and *z*, that satisfy the three original equations. The equations are "inconsistent."

A SYSTEM WITH INFINITELY MANY SOLUTIONS:

Consider now the equations:

$$\begin{aligned} x + 2y - z &= 4 \\ x + 3y + 2z &= 5 \\ y + 3z &= 1 \end{aligned}$$

Subtract the first equation from the second to obtain:

$$\begin{aligned} x + 2y - z &= 4 \\ y + 3z &= 1 \\ y + 3z &= 1 \end{aligned}$$

and subtract the second equation from the third:

$$\begin{aligned} x + 2y - z &= 4 \\ y + 3z &= 1 \\ 0 &= 0 \end{aligned}$$

This shows that the variable *z* can adopt any value. For example, if $z = 3$, then $y = 1 - 3z = -8$, and $x = 4 - 2y + z = 4 - 2(1 - 3z) + z = 2 + 7z = 23$, or if $z = 100$, then $y = -299$, and $x = 702$.

There are infinitely many solutions to this system of equations, each of the form:

$(x, y, z) = (2+7z, 1-3z, z)$. (Note that there is nothing special here about the variable "z." If we rewrote the equations so that the variables appeared in a different order, then we could present an infinite set of solutions with either *x* or *y* the focus variable.)

AN UNDETERMINED SYSTEM:

A system of equations possessing more variables than equations is called "underdetermined." For example, this month's puzzle was an underdetermined system: there were four variables – pig, mouse, cow, rooster – but only three pieces of information about them – the three row sums. It is possible that such a system might be inconsistent (for instance, there is clearly no solution to $x + y + z = 1$ and $2x + 2y + 2z = 503$), but if an underdetermined system has one solution, then it is guaranteed to have infinitely many solutions. For example, consider the system of the challenge problem:

$$\begin{aligned} 2A + B + C + D &= 80 \\ A + 2B + C + D &= 95 \\ 3A + 2B &= 80 \end{aligned}$$

Let's interchange the first and second rows:

$$\begin{aligned} A + 2B + C + D &= 95 \\ 2A + B + C + D &= 80 \\ 3A + 2B &= 80 \end{aligned}$$

Subtract two copies of the first equation from the second and three copies of the first equation from the third:

$$\begin{aligned} A + 2B + C + D &= 95 \\ -3B - C - D &= -110 \\ -4B - 3C - 3D &= -205 \end{aligned}$$

Despite the appearance of fractions, divide the last two equations by -3 and -4, respectively:

$$\begin{aligned} A + 2B + C + D &= 95 \\ B + \frac{1}{3}C + \frac{1}{3}D &= \frac{110}{3} \\ B + \frac{1}{4}C + \frac{1}{4}D &= \frac{205}{4} \end{aligned}$$

and then subtract the second equation from the third:

$$\begin{aligned} A + 2B + C + D &= 95 \\ B + \frac{1}{3}C + \frac{1}{3}D &= \frac{110}{3} \\ \frac{5}{12}C + \frac{5}{12}D &= \frac{175}{12} \end{aligned}$$

Finally, multiply the third equation through by $\frac{5}{12}$ to obtain:

$$\begin{aligned} A + 2B + C + D &= 95 \\ B + \frac{1}{3}C + \frac{1}{3}D &= \frac{110}{3} \\ C + D &= 35 \end{aligned}$$

It is possible to conduct a version of "back substitution" here. We have:

$$C + D = 35$$

and so

$$\begin{aligned} B &= \frac{110}{3} - \frac{1}{3}C - \frac{1}{3}D \\ &= \frac{110}{3} - \frac{35}{3} \\ &= 25 \end{aligned}$$

and

$$\begin{aligned} A &= 95 - 2B - C - D \\ &= 95 - 50 - 35 \\ &= 10 \end{aligned}$$

Although the values of *A* and *B* are determined, the values of *C* and *D* are not. All we know is that their sum must be 35. Thus, whatever value *D* adopts, we are required to set $C = 35 - D$. There are infinitely many solutions to this system all of the form: $(A, B, C, D) = (10, 25, 35 - D, D)$.

OVER-DETERMINED SYSTEMS:

A system of equations with more equations than variables is called "over-determined." It is possible that such a system might have just one solution, infinitely many solutions, or no solutions. For example, the system:

$$A + B = 2$$
$$A - B = 0$$
$$A - 2B = -1$$

has precisely one solution: $(A, B) = (1,1)$. The system:

$$x + y = 1$$
$$2x + 2y = 2$$
$$3x + 3y = 3$$

has infinitely many solutions (all of the form $(x,\ y) = (1 - y, y)$) and the system:

$$u + v = 2$$
$$u - v = 3$$
$$2u + 2v = 5$$
$$u - 2v = 7$$

has no solutions.

A FINAL QUESTION:

We have only presented systems of equations with either no solutions, precisely one solution, or infinitely many solutions. Explain why it is impossible for a system of linear equations to have just two solutions or just three solutions.