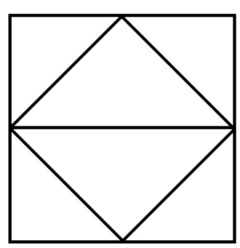
TAKING IT FURTHER

We have two outstanding questions:

1) Suppose each intersection of a street map possesses an even number of streets emanating from it. Is it guaranteed that there is a journey through the map that traverses each and every street precisely once, starting and ending at the same intersection?
2) Suppose a street map possesses precisely two intersections with an odd number of streets emanating from them. Is it guaranteed that there is a journey that traverses each and every street precisely once, starting and ending at the "odd intersections"?

For example, suppose we modify (c.) in the original puzzle so that instead of two horizontal lines slicing through the interior diamond, we have just one that runs from the left point of the diamond to the right point. Can this diagram be traced? How?



Notice that we are assuming that any street map we are given "comes in one piece," or, as mathematicians like to say, "connected." This means that it is always possible to reach any one intersection from another along a series of streets. (For example, a street map that covers two islands with no connecting bridge between them is not connected and cannot be traversed without lifting ones pencil from the page. One cannot walk the streets of both islands without a boat!)

The first person to consider these questions was Swiss mathematician Leonard Euler (1707-1783). He showed that the answer to each question is yes! Here's an outline of his explanations.

1. Suppose we are given a street map for which an even number of streets meet at each intersection. Select an arbitrary intersection (call it A) and begin a journey, traversing each street encountered only once. Wander along this journey for as long as possible, and consider where you might get "stuck." As every intersection possesses an even number of streets emanating from it, each time you enter an intersection different from A you are guaranteed an unused street by which to exit. Thus you will never be stuck at an intersection different from A, and your initial journey through the map must end you back at A.

   If by chance your journey traversed every street in the map, then we are done: you have just completed a journey that begins and ends at the same location traversing every street. In all likelihood, however, this initial journey will miss some streets on the map.

   Consider the collection of streets you did visit. Color them blue on the map and color all the remaining (missed) streets red. Notice that, by design, the blue streets come in “enter-exit” pairs at each intersection visited. Since the total number of streets at any intersection in the map is even, the number of red edges at any intersection must also be even. Thus the red edges, unto themselves, represent a street map with an even number of streets at each intersection. Moreover, as the original map is connected, there must be at least one intersection along the path of blue edges that has a red edge connected to it. Starting at this intersection, call it B, follow an arbitrary path of red edges until you get stuck. By precisely the same reasoning as above, you can only be stuck at the same intersection you started, and this is a fortunate occurrence for we can now revisit our initial journey and adjoin our new loop of red streets to it. To see this, go back to intersection A and follow the path of blue streets to B. Now complete the small loop of red streets to return to B and then continue the path of blue streets from B back to A. If we now color the red streets you used blue as well, we now have a bigger loop of blue streets starting and ending at A and fewer “unused” red streets.

We can now repeat the process of the previous paragraph yet again, finding a small loop of red streets to addend to this blue loop, and do this multiple times until every red street is colored blue. The final result is a single loop of streets, starting and ending at A that traverses each and every street precisely once!

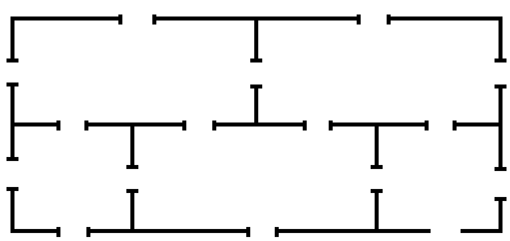
2. Fortunately, we have now completed all the hard work. To quickly resolve the second issue, draw a false street between the two odd intersections. (You might have to arch up into the third dimension to do this!) This produces a street map

with all intersections even. By the previous discussion, there is a loop of streets that traverses each and every street precisely once, including the false street. Now delete this false addition. What remains is a path from one end of the faux street to the other that represents a solution to the original question!

<u>ANOTHER VARIATION</u>:

Here's a classic puzzle:

> *Here's the floor plan of a house with doorways leading between rooms and from rooms to the outside.*



> *Is it possible to start at some location (either in a room or outside) and walk through every doorway precisely once?*

Can you see how to use the tools and techniques we have developed thus far to prove that this puzzle is unsolvable?

Challenge: Draw this house plan on the surface of a donut. Show that the puzzle can now be solved!

FINAL THOUGHT:

Did you notice that in each street map we have considered that the count of "odd intersections" was even? Try to construct an example of a street map with precisely one odd intersection. Can it be done? How about a street map with three, or five, or any odd number of odd intersections?

FURTHER READING:

Mathematicians prefer to call any diagram of streets and intersections a "graph" and call the study of properties of graphs "graph theory." There is certainly plenty of material in texts and on the web about this fascinating subject. Two particularly accessible pieces are Gary Chartrand's wonderful book *Introductory Graph Theory* (Dover, 1977) and the opening chapter of *For All Practical Purposes: Mathematical Literacy in Today's World* (5^{th} edition) produced by COMAP, the Consortium for Mathematics and its Applications (and published by W. H. Freeman and Company, 2000).