# $\mathscr{I}^m_\infty$ PROBLEMS SHEET

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1. Exercise. — Let $P : \mathbb{R}^2 \to \mathbb{R} : (x_1, x_2) \mapsto x_1$ be the projection on the horizontal axis.

(A) Try to prove that if $B \subseteq \mathbb{R}^2$ is Borel then so is $P(B)$ (see e.g. H. Lebesgue *J. Math. Pures Appl.* 1905 for a famous attempt). Given an arbitrary sequence $\langle A_j \rangle_{j \in \mathbb{N}}$ of subsets of $\mathbb{R}^2$, are the sets $P(\cap_{j\in\mathbb{N}} A_j)$ and $\cap_{j\in\mathbb{N}} P(A_j)$ equal ? What about when $\langle A_j \rangle_{j \in \mathbb{N}}$ is nonincreasing ?

(B) Find a Borel set $B \subseteq \mathbb{R}^2$ such that $P(B)$ is not Borel (hint : make your way through A. Kechris *Descriptive set theory* or the chapter on "analytic sets" in D. Cohn *Measure Theory*).

2. Exercise. — This is about the usual integral geometric measure $\mathscr{I}^1_1$ in $\mathbb{R}^2$.

(A) Let $a, b \in \mathbb{R}^2$ and define the line segment $[\![a,b]\!] = \mathbb{R}^2 \cap \{a + t(b-a) : 0 \leqslant t \leqslant 1\}$. Compute $\mathscr{I}^1_1([\![a,b]\!])$.

(B) Compute $\mathscr{I}^1_1(U)$ for any nonempty open set $U \subseteq \mathbb{R}^2$.

(C) Open ended : Find about Buffon's needle and Monte-Carlo methods to compute $\pi$ and how they relate to the ideas behind the definition of $\mathscr{I}^1_1$.

3. Exercise. — This exercise suggests one way of proving Theorem 1.1. Given $A \subseteq \mathbb{R}^n$ define

$$N_A : \mathbb{G}(n,m) \times \mathbb{R}^n \to \mathbb{R}^+ \cup \{\infty\} : (W, y) \mapsto \operatorname{card} A \cap P_W^{-1}\{y\}$$

and

$$\boldsymbol{\chi}(B) = \mathbb{G}(n,m) \times \mathbb{R}^n \cap \{(W,y) : y \in P_W(B)\}\,.$$

(A) Show that if $B$ is a Borel set then $\boldsymbol{\chi}(B)$ is universally measurable (hint : use some results from descriptive set theory e.g. in D. Cohn ibid.)

(B) If $B_1 \subseteq B_2$ then $\boldsymbol{\chi}(B_1) \subseteq \boldsymbol{\chi}(B_2)$.

(C) Assume $\langle \mathscr{B}_j \rangle_{j\in\mathbb{N}}$ is a sequence of partitions of $A$ such that

(i) $\mathscr{B}_{j+1}$ is a refinement of $\mathscr{B}_j$ ;

ii) $\lim_j \sup\{\operatorname{diam} B : B \in \mathscr{B}_j\} = 0$.

Establish that the sequence $\langle \sum_{B\in\mathscr{B}_j} \mathbb{1}_{\boldsymbol{\chi}(B)} \rangle_{j\in\mathbb{N}}$ is nondecreasing and that

$$N_A = \lim_j \sum_{B\in\mathscr{B}_j} \mathbb{1}_{\boldsymbol{\chi}(B)}\,.$$

(D) Apply the monotone convergence theorem.

4. Exercise. — This is about the integral geometric measure $\mathscr{I}^m_\infty$ in $\mathbb{R}^n$.

(A) Let $a, b \in \mathbb{R}^2$. Compute $\mathscr{I}^1_\infty([\![a,b]\!])$.

(B) Compute $\mathscr{I}^1_\infty([a,b] \times [a,b])$, and next $\mathscr{I}^1_\infty(U)$ where $\emptyset \neq U \subseteq \mathbb{R}^2$ is open.

(C) Is $\mathscr{I}^m_\infty$ invariant under isometries ? How does it behave under homotheties ?

(D) Where does the proof of Theorem 1.1 fail when $\mathscr{I}_1^m$ is replaced with $\mathscr{I}_\infty^m$? Prove the following weaker version : If $B \subseteq \mathbb{R}^n$ is Borel then

$$\left\| \int_\bullet \operatorname{card}(B \cap P_\bullet^{-1}\{y\}) d\mathscr{H}^m(y) \right\|_{L_\infty(\mathbb{G}(n,m),\gamma_{n,m})} \leqslant \mathscr{I}_\infty^m(B) .$$

5. Exercise. — This is an informal discussion about the relation between area and triangulations (or polyhedral approximation). If $f : [0,1] \to \mathbb{R}^n$ is an injective Lipschitz curve then

$$\int_0^1 |f'(t)| d\mathscr{L}^1(t) = \sup \left\{ \sum_{k=1}^{\kappa} |f(t_k) - f(t_{k-1})| : 0 = t_0 < t_1 < \ldots < t_\kappa = 1 \right\} \quad (1)$$

and for each such subdivision $0 = t_0 < t_1 < \ldots < t_\kappa = 1$ and each $k = 1, \ldots, \kappa$ one has

$$[\![ f(t_{k-1}), f(t_k) ]\!] \subseteq P_{L_k}(f([t_{k-1}, t_k])) \quad (2)$$

where $L_k$ is the (affine) line containing $f(t_{k-1})$ and $f(t_k)$.

(A) Prove both facts if you haven't seen a proof of these before.

(B) If $m > 1$ then there is no analog of (1) (hint : google `Hermann Schwarz lantern` or draw an accordeon — and think).

(C) The unpleasant phenomenon illustrated by Hermann Schwarz' example can be avoided by using *regular* triangulations. Specifically (say $n = 3$ and $m = 2$), if $T \subseteq \mathbb{R}^2$ is a triangle and $f : T \to \mathbb{R}^3$ is (say injective and) Lipschitz then

$$\mathscr{H}^2(f(T)) = \lim_j \sum_{S \in \mathscr{S}_j} \mathscr{H}^2([\![ f(v_1(S)), f(v_2(S)), f(v_3(S)) ]\!])$$

for any sequence of triangulations $\langle \mathscr{S}_j \rangle_{j \in \mathbb{N}}$ of $T$ such that

(i) There exists $\theta > 0$ such that for every $j \in \mathbb{N}$ and every triangle $S \in \mathscr{S}_j$, each angle of $S$ is bounded below by $\theta$ ;

(ii) $\lim_j \sup\{\operatorname{diam} S : S \in \mathscr{S}_j\} = 0$.

In the above formula we have used the following notations : (1) $v_i(S)$, $i = 1, 2, 3$, denote the vertices of a triangle $S$ ; (2) $[\![ a, b, c ]\!]$ is the triangle in $\mathbb{R}^3$ with vertices $a, b, c \in \mathbb{R}^3$, i.e. the convex envelope of $\{a, b, c\}$.

(D) What about (2) if $m > 1$ ? Show by an example that the inclusion need not hold in general, but always "almost" holds (a proper usage of "almost" in this sentence would be one that guarantees an appropriate modification of the proof of Theorem 2.1 works for $n = 3$, $m = 2$).

6. Exercise. — This is about the Cantor square $S \subseteq \mathbb{R}^2$ introduced in class. All notations are consistent with those used in class.

(A) Check that if $D \subseteq \mathbb{R}^2$ denotes the set of vertices of the squares used in the construction, then $D \subseteq S$, $D$ is countable, and $P_L \restriction (S \setminus D)$ is injective.

(B) Write $\tilde{S} = S \setminus D$ and $u = P_{L^\perp} \circ (P_L \restriction \tilde{S})^{-1}$. Check that $(P_L \restriction \tilde{S})^{-1}$ is continuous (but of course its domain $P_L(\tilde{S})$ is a completely disconnected Borel set ; that $u$ be Borel follows by general descriptive set theory arguments, but in this instance there is an easier proof of the stronger result that $u$ is continuous).

(C) Show that for every $\varepsilon > 0$ there exists a continuous curve $\Gamma \subseteq \mathbb{R}^2$ such that $\mathscr{H}^1(S \setminus \Gamma) < \varepsilon$ (hint : Use the fact that $\mathscr{H}^1 \llcorner S$ is Radon, and Tietze's extension theorem).

(D) Try to visualize such a curve $\Gamma$ and think about whether you can estimate the oscillation of $u \restriction C$ where $C \subseteq P_L(\tilde{S})$ is any compact set so that $u \restriction C$ is continuous and $\mathscr{H}^1(C) > 0$.

7. Exercise. — This is about Lipschitz graphs.

(A) Prove the bow-tie lemma and establish how $s$ and $\operatorname{Lip} f$ relate.

(B) *(Tilt your head)* If $G \subseteq \mathbb{R}^n$ is a Lipschitz graph over a subset of $W \in \mathbb{G}(n,m)$ then there exists $\varepsilon > 0$ such that for every $W' \in \mathbb{G}(n,m)$ with $d(W,W') < \varepsilon$, $G$ is a also a Lipschitz graph over a subset of $W'$. Here $d(W,W') = \|P_W - P_{W'}\|$ (Hilbert-Schmidt norm for instance). How do $\varepsilon$ and $\operatorname{Lip} f$ relate ?

(C) Let $u : [0,1] \to [0,1]$ be the Cantor function (also called the devil's staircase). Since $u$ is nondecreasing, its graph $G$ has finite length, i.e. $\mathscr{H}^1(G) < \infty$ (prove the general fact). Since $G$ is a continuous curve, it is consequently also a Lipschitz curve (prove the general fact). However $u$ is not Lipschitz (check it). Is $G$ a Lipschitz graph over some line $W \in \mathbb{G}(2,1)$ ?

(D) If $u : [0,1] \to \mathbb{R}$ is continuous then its graph is not purely $(\mathscr{H}^1, 1)$-unrectifiable (hint : T.C. O'Neil, *Real Anal. Exch.*, 2000). Why does this not contradict Theorem 3.1 about the "parametrization" of a large part of the Cantor square ?

8. Exercise. — Let $S \subseteq \mathbb{R}^2$ be the Cantor square. Show that for every $x \in S$

$$\Theta^1_*(\mathscr{H}^1 \llcorner S, x) < 1 \,.$$

(hint : Draw a picture). This provides another proof that $S$ is purely $(\mathscr{H}^1, 1)$-unrectifiable.