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MSRI Summer School Lecture #3: Total positivity for the Grassmannian

References: Postnikov preprint (arXiv 2006),
J. Scott "Grassmannians + Cluster Algebras"

Def: The Grassmannian $Gr_{k,n}(\mathbb{R}) = \{V : V \subseteq \mathbb{R}^n, \dim V = k\}$
vector space

Note: Can represent each element of $Gr_{k,n}(\mathbb{R})$ as a full rank $k \times n$ matrix A.
Rows of A span k-dim'l subspace of $\mathbb{R}^n$.
$A \sim A'$ if span the same subspace.

So: $Gr_{k,n}(\mathbb{R}) = Gl_k \backslash \{\text{Full rank } k \times n \text{ matrices}\}$

Given a $k \times n$ matrix A, and $I \in \binom{[n]}{k}$, let $\Delta_I(A)$ denote the $(k \times k)$ minor of A which uses column set I. (Plucker coordinate)
We have Plucker embedding
$$Gr_{kn} \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$$
$$A \longmapsto (\Delta_I(A))_{I \in \binom{[n]}{k}}$$

Def: The TNN Grassmannian $(Gr_{kn})_{\geq 0}$ (resp the TP " " $(Gr_{kn})_{>0}$) is the subset of Gr_{kn} that can be represented by full rank $k \times n$ matrices A s.t. all $\Delta_I(A) \geq 0$ (resp > 0).

Ex: Let $A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{R}$

Then $A \in Gr_{k,n}(\mathbb{R})$.

In order for A to lie in $(Gr_{k,n})_{>0}$, need

$\Delta_{12}(A) = 1 > 0$ $\quad$ $\Delta_{13}(A) = c > 0$

$\Delta_{14}(A) = d > 0$ $\quad$ $\Delta_{23}(A) = -a > 0$

$\Delta_{24}(A) = -b > 0$ $\quad$ $\Delta_{34}(A) = ad - bc > 0.$

Note: The 2×2 minors of A are not all indep. They satisfy a Plucker relation:

$$\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$$

$$c \cdot (-b) = 1 \cdot (ad-bc) + d \cdot (-a)$$

More generally, if $A \in Gr_{2,n}$, then the minors of A satisfy the 3-term Plucker relations:

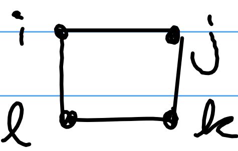
for any $i<j<k<l$,

$$\Delta_{ik}\Delta_{jl} = \Delta_{ij}\Delta_{kl} + \Delta_{il}\Delta_{jk}$$

And if $A \in Gr_{m,n}$, then the minors satisfy:

for any $i<j<k<l$ and $(m-2)$ subset I of $\{1,\dots,n\}$ disjoint from i,j,k,l:

$$\Delta_{I\cup\{i,k\}}\Delta_{I\cup\{j,l\}} = \Delta_{I\cup\{i,j\}}\Delta_{I\cup\{k,l\}} + \Delta_{I\cup\{i,l\}}\Delta_{I\cup\{j,k\}}$$

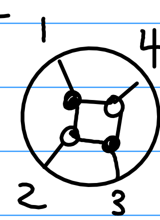
Mnemonic 

Q: How many minors does one need to test (& which minors) to determine if some $A \in Gr_{kn}(\mathbb{R})$ lies in $(Gr_{kn})_{>0}$?

Recall: When we were studying $(SL_n)_{>0}$, we used the combinatorics of double wiring diagrams to read off sets of chamber minors. Those sets of minors were positivity tests for $(SL_n)_{>0}$.

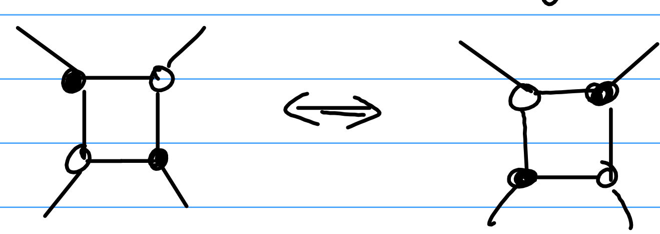
Goal: For $(Gr_{kn})_{>0}$, one can do something similar, using plabic graphs.
(Sort of like double wiring diagrams on a circle.)

Def: A plabic graph is a graph embedded in a disk w/ n boundary vertices labeled $1, 2, \dots, n$ counterclockwise, s.t. that internal vertices are colored black or white, and boundary vertices have degree 0 or 1.
Disallow: non-boundary leaves, isolated components.

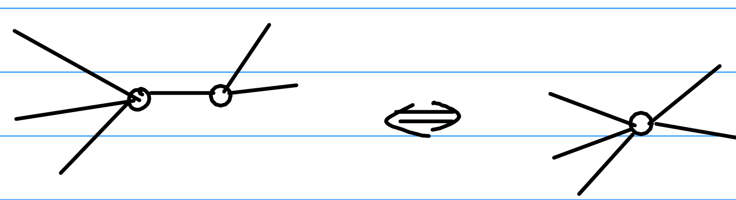


We will define some moves in order to define equivalence classes of plabic graphs:

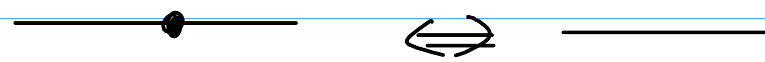
(M1) Square move



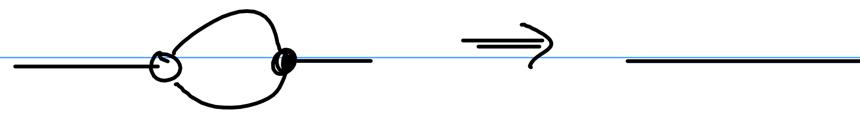
(M2) Unicolored edge (un)contraction



(M3) Middle vertex insertion / removal



(R1) Parallel edge reduction



<u>Def:</u> Two plabic graphs are <u>move-equivalent</u> if they can be obtained from each other using (M1), (M2), (M3)

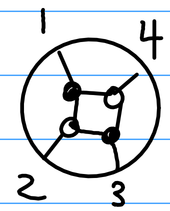
<u>Def:</u> A plabic graph is called <u>reduced</u> if there is no graph in its move equivalence class to which one can apply a reduction.

Analogy: reduced plabic graph ~ reduced decomposition of a permutation

<u>Now:</u> we can assign a permutation to every reduced plabic graph G.
Can also label regions of G by k-element subsets of $\{1, \ldots, n\}$.

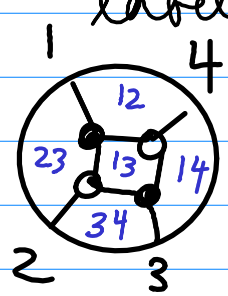
Rules of the road:
turn right at a black vertex & left at a white vertex.

Def/thm: Let G be reduced plabic graph w/ n bdy vertices. For each bdy vertex i, follow the rules of the road: this produces a trip T_i from vertex i to some vertex $\pi(i)$. Then π is a permutation in S_n, called the trip permutation of G.

Ex: Let $G =$ 

Then $\pi = \begin{matrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 1 & 2 \end{matrix}$

Algorithm: Each trip T_i partitions the regions of G into 2 subsets, those on the left & those on the right of G. Put an i in all regions to the left of T_i. After doing this for all $i \in \{1, \dots, n\}$, every region will contain the same number of labels.

Ex: 

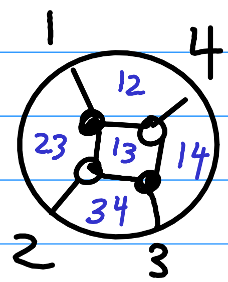
Def: Let $\pi_{k,n}$ denote the permutation

$$\begin{array}{cccccccccc} 1 & 2 & 3 & \ldots & k-1 & k & k+1 & \ldots & n \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow \\ n-k+1 & n-k+2 & n-k+3 & & n-1 & n & 1 & \ldots & n-k \end{array}$$

Theorem: Let G be a reduced plabic graph whose trip permutation is $\pi_{k,n}$. Let $S = \{\Delta_I : I \text{ labels a region of } G\}$. Then S is a positivity test for $(Gr_{k,n})_{>0}$. That is, an element A of $Gr_{k,n}(\mathbb{R})$ is in $(Gr_{k,n})_{>0}$ iff $\Delta_I(A) > 0$ for each $I \in S$.

Rk: $\mathbb{C}[Gr_{k,n}]$ has structure of cluster algebra (Scott); the sets S comprise some of the clusters.

Ex:

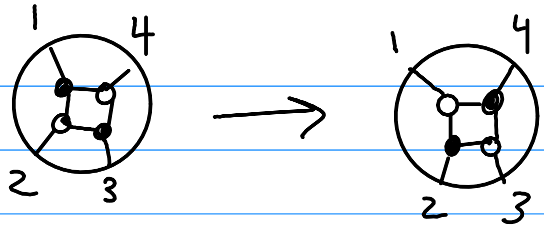


Trip perm is $\pi_{2,4}$.

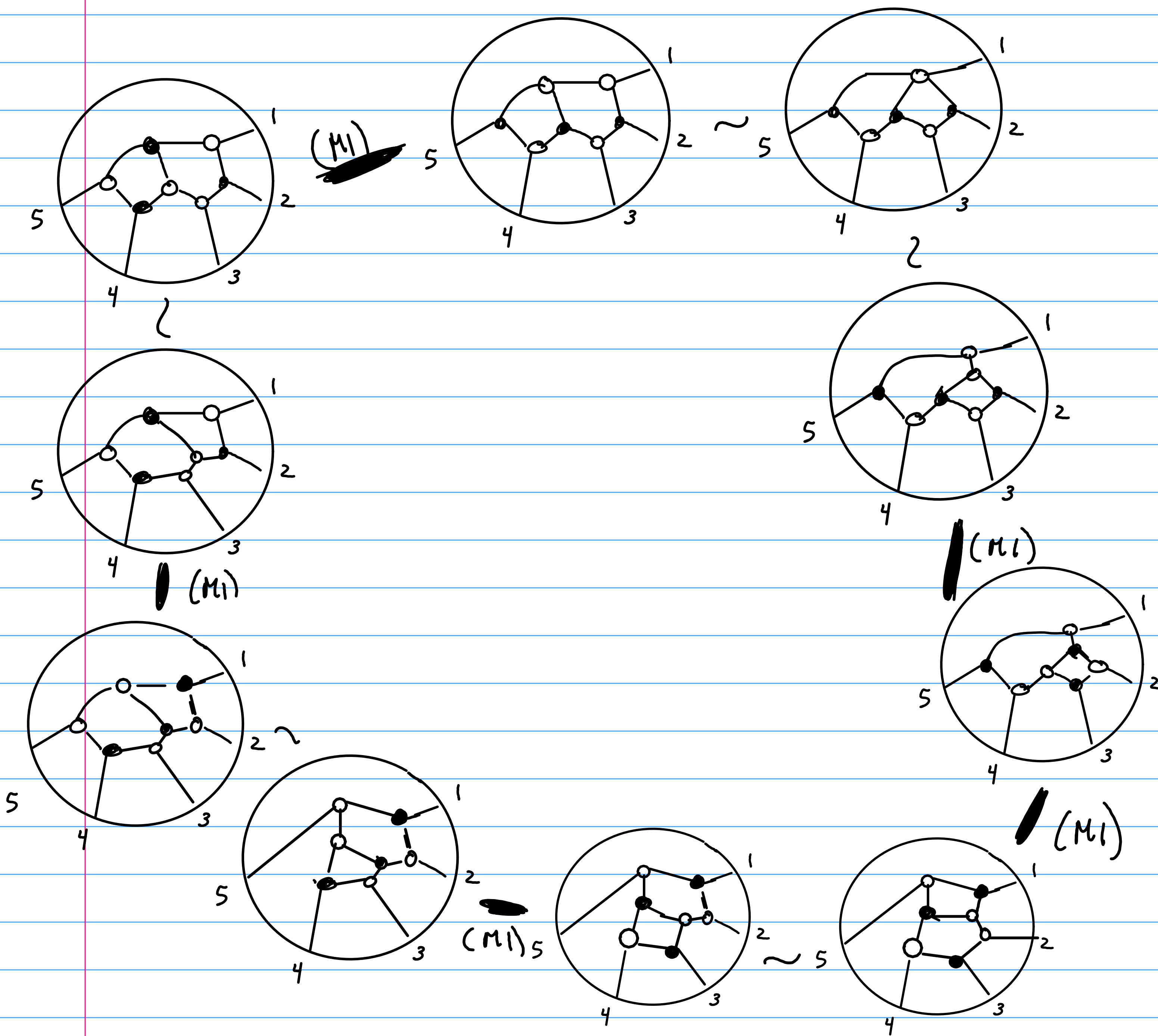
$\therefore S = \{\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{14}, \Delta_{13}\}$ is a positivity test for $(Gr_{2,4})_{>0}$.

Can prove this, since $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$!

How do we find more reduced plabic graphs of type $\pi_{k,n}$? Apply the moves M1, M2, M3.



More complicated example:



Now: What can we say about elements of the TNN Grassmannian (not necessarily in TP part)?

Let $M \subseteq \binom{[n]}{k}$.
The positroid cell S_M^{tnn} is the subset of elements A of $(Gr_{kn})_{\geq 0}$ s.t. $\Delta_I(A) > 0$ iff $I \in M$.

Thm (Postnikov): Either S_M^{tnn} is empty or S_M^{tnn} is a cell, ie. is homeomorphic to an open ball.

One would like to classify the cells of S_M^{tnn}, + answer questions like what are their dimensions, when is one contained in closure of the other, what is the homotopy type of $(Gr_{kn})_{\geq 0}$, etc...

Thm (P): The cells of $(Gr_{kn})_{\geq 0}$ are naturally labeled by (& in bijection with):

(1) • Ⅎ-diagrams of type (k,n).
(2) • Decorated permutations π on n letters w/ k weak excedances
(3) • equivalence classes of reduced plabic graphs w/ trip perm π

⋮

Def: a ⅃-diagram of type (k,n) is a Young diagram contained in a $k \times (n-k)$ rectangle where boxes filled w/ 0's and +'s s.t. the following pattern is forbidden

$$\begin{array}{cc} & + \\ & \vdots \\ + \cdots & 0 \end{array}$$

Ex:

+	+	0	0	+
+	+	+	+	
0	0	0	+	

Dimension of cell = number of +'s in ⅃-diagram.

Def: A decorated permutation on n letters is a permutation where each fixed point is decorated w/ the label "clockwise" or "counterclockwise."

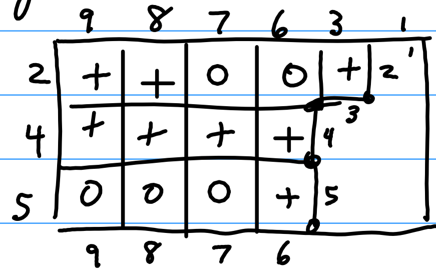
A <u>weak</u> <u>exceedance</u> of a dec perm π is a position i s.t. $\pi(i) > i$ <u>OR</u>

$\pi(i) = i$ and this is <u>counterclockwise</u>.

Thm: Cells of $(Gr_{kn})_{\geq 0}$ in bijection w/ dec perms on n letters w/ k weak exceedances.

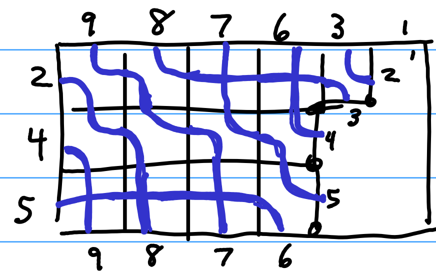
Bijection from ⅃-diagrams of type (k,n) to dec perms of type (k,n) (in S_n w/ k weak exceedances)

Label border:



Replace + with

and 0 with +



Follow the pipes from northwest border to southeast border to get a permutation:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 8 & 2 & 9 & 6 & 4 & 5 & 3 & 7 \end{matrix}$$

Consider a fixed point labeling a column (resp row) to be clockwise (resp counterclockwise).

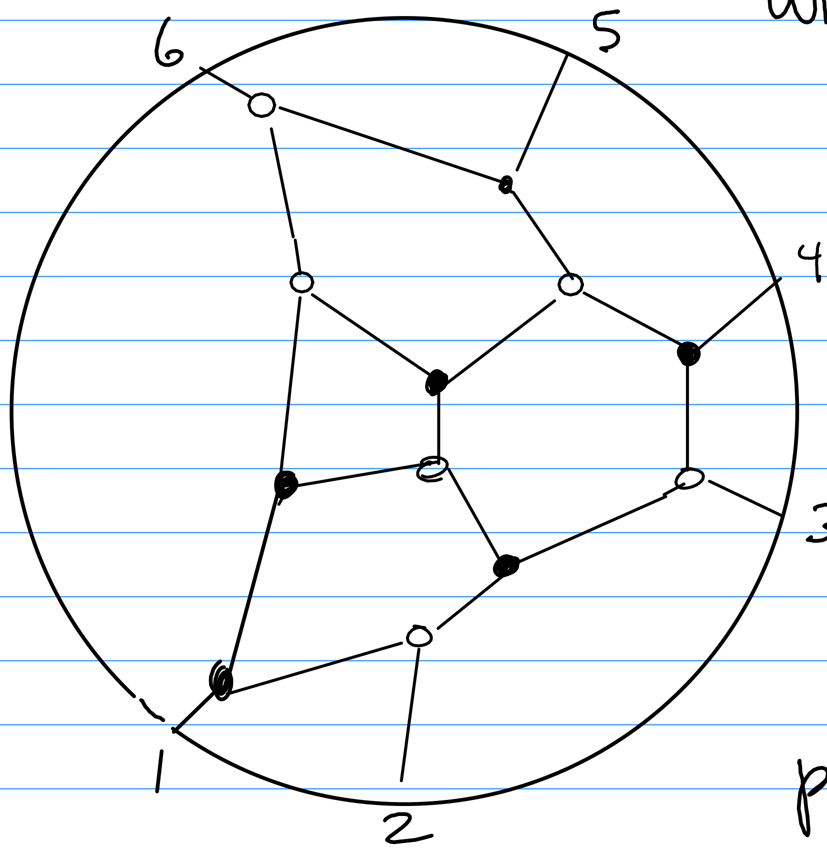
$\therefore$ this dec perm is $\pi = (\overset{\text{clock}}{1}; 8, 2, 9, 6, 4, 5, 3, 7)$

<u>Note</u>: Has 3 weak exceedances

Exercises

1. Check that if a reduced plabic graph G has trip perm π, & we apply a <u>move</u>, the trip perm will be preserved.

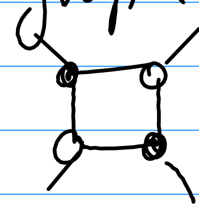
2. Let G be the following red plabic graph.



What is the trip perm? Use G to find a positivity test for $(Gr_{3,6})_{>0}$.

Now apply moves to find some more positivity tests.

3. Consider a reduced plabic graph G, w/ a local configuration



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Apply move M1.

Can you write down an algebraic relation relating the 2 corresponding positivity tests?

4. Draw all the ⅃-diagrams for cells of $(Gr_{2,4})_{\geq 0}$

5. Consider $Gr_{2,n}$. Draw an n-gon w/ vertices labeled $1, \ldots, n$ in order.
We can identify the minors Δ_{ij} w/ diagonals + edges of the n-gon.
Find an algorithm to associate a reduced plabic graph w/ perm $\pi_{2,n}$ to each triangulation of an n-gon.