Lauren Williams

MSRI School Lecture #4: Tropical Geometry

Reference: Book by Diane Maclagan + Bernd Sturmfels

"Def:" Trop geom is alg geom, where instead of working over $\mathbb{C}$ (or another field) we work over the **tropical semiring** $(\mathbb{R}, \oplus, \otimes)$.

Here $a \oplus b := \min(a, b)$ and
$a \otimes b := a + b$

Ex: $2 \otimes (1 \oplus 4) = 2 + \min(1, 4) = 3$

Note: $\min(a, \infty) = a$, so we can work over $\mathbb{R} \cup \infty$ and consider ∞ as the additive identity. Then $(\mathbb{R} \cup \infty, \oplus, \otimes)$ is a semiring — assoc, dist, etc, but no additive inverse.

In alg geom, we work with polynomials. In trop geom, we "tropicalize" these polys, which turns them into piecewise linear functions.

Ex: $f(x,y,z) = x^3 y + y^2 z + x^2 + 1$
$\mathrm{trop}(f) = \min\{3x + y, 2y + z, 2x, 0)$

I'll explain carefully in a few min.

In alg geom, we study varieties. To a first approximation, a variety is the set of common zeros of polynomial equations.

Tropically, this corresponds to taking the <u>nonlinear</u> <u>locus</u> in $\mathbb{R}^2$ of the polynomial trop (f).

<u>Ex:</u> Let $f(x,y) = x+y+1$.
The variety $V(f)$ of f is

$$\{(x,y) \in \mathbb{C}^2 : x+y+1=0\}$$

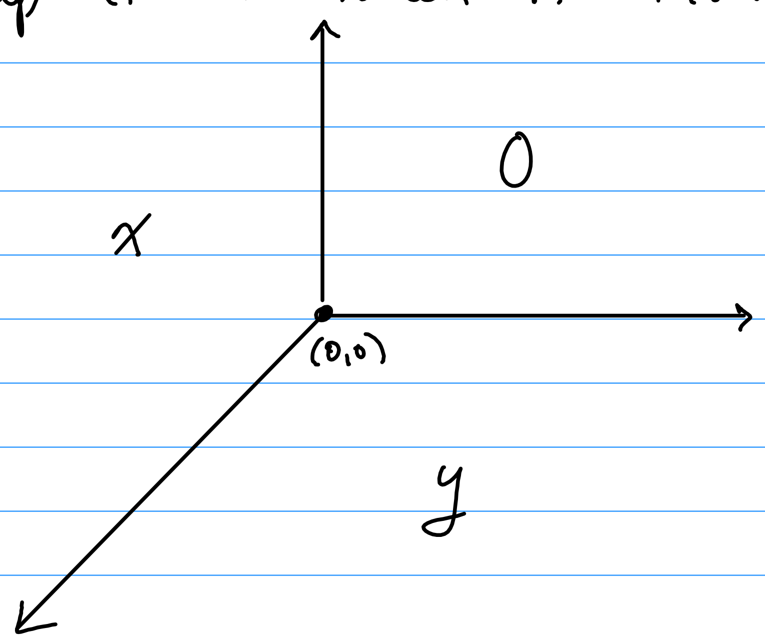
This is a <u>line</u> in $\mathbb{C}^2$.

$\mathrm{Trop}(f) = \min(x,y,0)$.
What is the <u>nonlinear</u> <u>locus</u> of $\mathrm{Trop}(f)$?

<u>Note:</u> ① If $x>0$ and $y>0$, $\mathrm{Trop}(f)(x,y) = 0$, so $\mathrm{Trop}(f)$ is <u>linear</u> (constant) in this region of $\mathbb{R}^2$.

② If $x<0$ and $y>x$, $\mathrm{Trop}(f)(x,y) = x$, so $\mathrm{Trop}(f)$ is <u>linear</u> in this region.

③ If $y<0$ and $x>y$, $\mathrm{Trop}(f)(x,y) = y$, so $\mathrm{Trop}(f)$ is linear in this region.



(also called corner locus)

- The nonlinear locus of Trop(f) is at the boundaries separating 2 of these 3 regions.
- Where the min is achieved by at least 2 of the terms in $\min(x, y, 0)$.
- This is the piecewise linear curve

$$= \{x = y \leq 0\} \cup \{x = 0 \leq y\} \cup \{y = 0 \leq x\}$$

Motivation: Why study tropical geometry?

- Polyhedral geom. is often easier than alg geom.
- Many invariants of a usual variety translate into invariants of the trop variety.
  E.g. Dimension is preserved under tropicalization. When we studied $f(x,y) = x + y + 1$, both the usual variety & the trop variety (nonlinear locus of Trop(f)) had dimension 1.
- One of first successful applications of trop geom to enum geom was work of Mikhalkin — counting the number of rational curves in $\mathbb{P}^2$ of a given degree d passing through a fixed set of points in gen. position.
  Translated problem in alg geom into problem in trop geom ⇝ combinatorics.

Now: more careful def of tropical hypersurface & variety.

Recall that if $f \in \mathbb{C}[x_1,\dots,x_n]$, the hypersurface
$V(f) := \{(a_1,\dots,a_n) \in \mathbb{C}^n \mid f(a_1,\dots,a_n) = 0\}$.

And if $I \subset \mathbb{C}[x_1,\dots,x_n]$ is an ideal, the variety $V(I) := \{(a_1,\dots,a_n) \in \mathbb{C}^n \mid f(a_1,\dots,a_n) = 0 \ \forall f \in I\}$

$$= \bigcap_{f \in I} V(f)$$

To introduce the tropical versions of these objects, need to work w/ Puisseux series.

Let $K = \mathbb{C}\{\{t\}\} = \bigcup_{n \geq 1} \mathbb{C}((t^{1/n}))$

be the Puiseux series.

Here $\mathbb{C}((t^{1/n}))$ means the ring of Laurent series in the variable $t^{1/n}$.

Each element $a \in K$ has the form
$$a = \sum_{q \in \mathbb{Q}} a_q t^q \quad \text{where}$$
$\{q \in \mathbb{Q} : a_q \neq 0\}$ is bounded below + has a common denominator.

Let $K^* = K \setminus \{0\}$. We define a valuation map
$$\text{val} : K^* \to \mathbb{R} \quad \text{by}$$
$$\text{val}(a) = \min\{q : a_q \neq 0\}$$

Note: $\mathrm{val}(a) = 0$ for any $a \in \mathbb{C}$.

Def: If $f \in K[x_1, \dots, x_n]$, the hypersurface $V(f) := \{(a_1, \dots, a_n) \in K^n \mid f(a_1, \dots, a_n) = 0\}$.

And if $I \subset K[x_1, \dots, x_n]$ is an ideal, the variety $V(I) := \{(a_1, \dots, a_n) \in K^n \mid f(a_1, \dots, a_n) = 0 \ \forall f \in I\}$

$$= \bigcap_{f \in I} V(f)$$

The tropical hypersurface $\mathrm{trop}\, V(f)$ is

$\mathrm{val}\left(V(f) \cap (K^*)^n\right) \subset \mathbb{R}^n$ and

use K^* because val only defined on K^*

the tropical variety $\mathrm{trop}\, V(I)$ is

$$\mathrm{val}\left(V(I) \cap (K^*)^n\right) = \bigcap_{f \in I} \mathrm{trop}\, V(f) \subset \mathbb{R}^n$$

We'd like a more concrete description of these objects.

Def: Let $S = K[x_1, \dots, x_n]$ and write $f \in S$ as

$$f = \sum_{u \in \mathbb{N}^n} c_u x^u \quad \text{where } x^u := \prod_{i=1}^n x_i^{u_i}.$$

Then $$\mathrm{trop}(f) = \min_{u \in \mathbb{N}^n,\ c_u \neq 0} \left(\mathrm{val}(c_u) + \sum_{i=1}^n u_i x_i\right),$$

Theorem: $\mathrm{trop}(V(f)) =$
$\left\{ w \in \mathbb{R}^n : \text{the minimum in the definition of } \mathrm{trop}(f)(w) \text{ is achieved at least twice} \right\}$.

Properties of tropical varieties
(Bergman '70, Bieri-Groves '84, Speyer-Sturmfels '02, Bogart-Jenson-Speyer-Sturm-Thomas '05)

(1) Let I be homog. prime ideal. Then trop $V(I)$ has same dim as $V(I)$.

(2) trop $V(I)$ is a polyhedral complex (or fan, depending on your def'n)
Is pure dim'l & connected in codim 1
Can go from one max'l face to another by going through ridges.

Let's compute some tropical hypersurfaces.

Ex 1: Let $f = tx^2 + 2xy + 3ty^2 + 5x + 7y - (t^2 + t^5)$.
Then $\mathrm{trop}(f) =$
$\min(2x+1,\ x+y,\ 2y+1,\ x,\ y,\ 2)$.

To draw $\mathrm{trop}(V(f))$, we determine which are the regions where each of the 6 terms is minimal.

1. When is 2 min'l? That is,

When is (a) $2 \le 2x+1$, (b) $2 \le x+y$, (c) $2 \le 2y+1$, (d) $2 \le x$, (e) $2 \le y$?

Note: (d) $\Rightarrow$ (a) and (e) $\Rightarrow$ (c)

(d) and (e) $\Rightarrow$ (b).

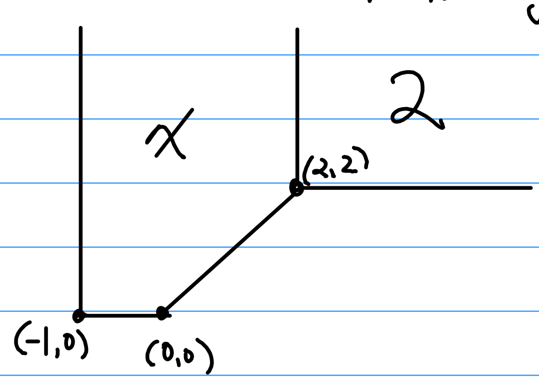
$\therefore$ 2 is minimal in region $x \ge 2$ and $y \ge 2$.

2. When is x min'l?

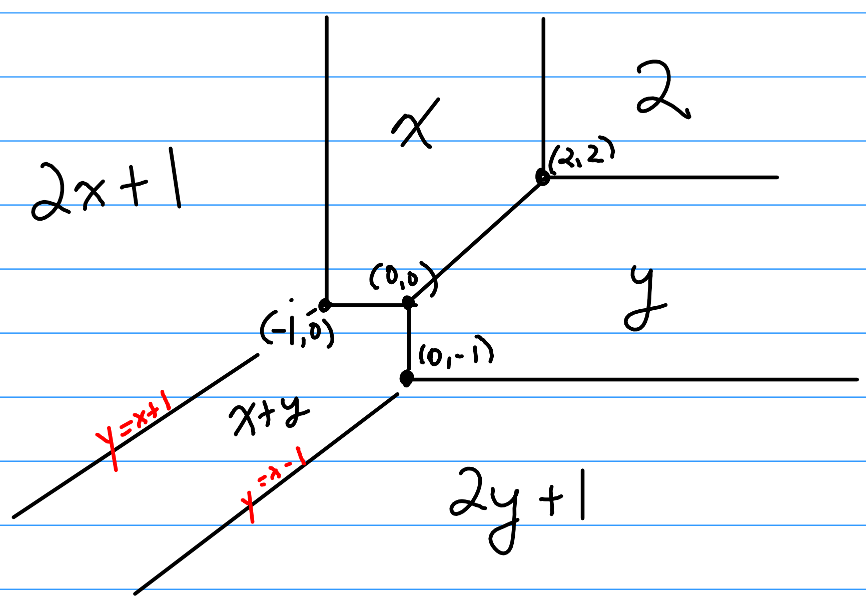
When is (a) $x \le 2x+1$ (b) $x \le x+y$, (c) $x \le 2y+1$, (d) $x \le y$, (e) $x \le 2$?

i.e. $x \ge -1$, $y \ge 0$, $x \le y$, $x \le 2$

Note: $y \ge 0$ and $x \le y \Rightarrow x \le 2y+1$



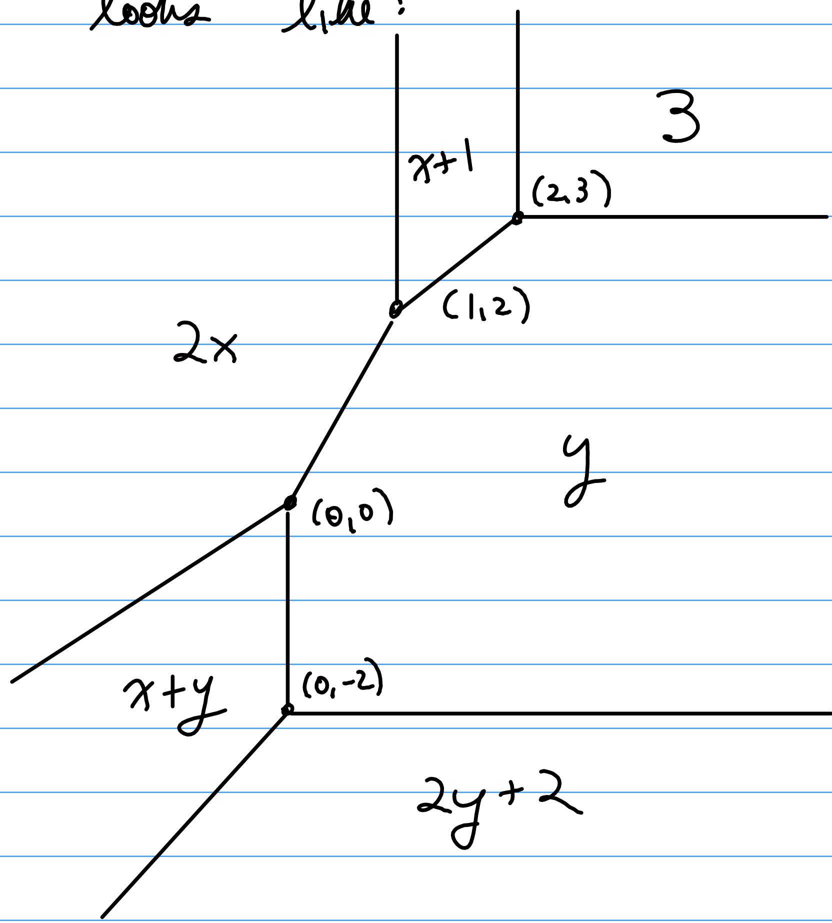
If we continue this process, we will get



Ex 2: Let $g = (t^2 - t^{5/2})y^2 + 5x^2 - 7xy + 8y - tx + t^3$.

$\mathrm{trop}(g) = \min(2+2y,\ 2x,\ x+y,\ y,\ x+1,\ 3)$

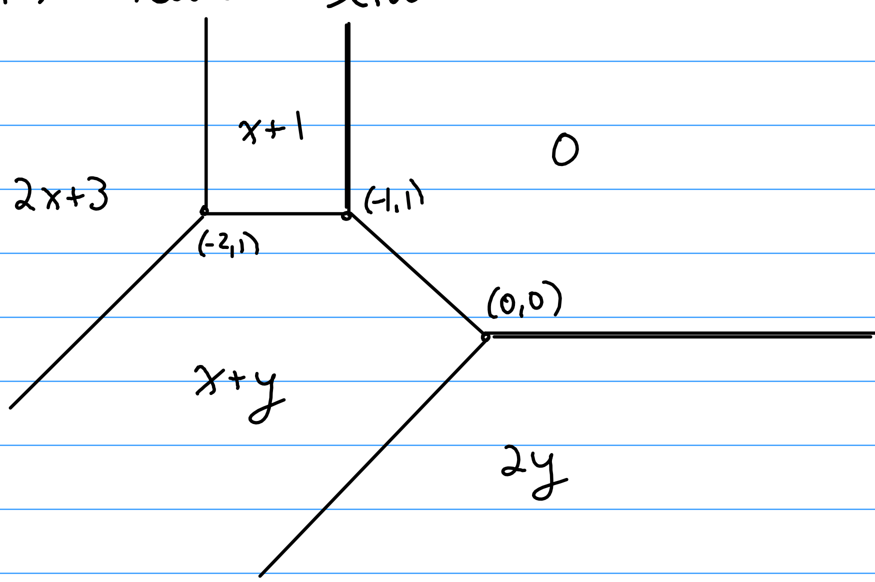
$\mathrm{Trop}(V(g))$ looks like:



Ex 3: Let $h = t^3x^2 - 7tx + 8xy - 7y^2 + 6$. Then

$\mathrm{trop}(h) = \min(2x+3,\ x+1,\ x+y,\ 2y,\ 0)$.

$\mathrm{trop}(V(h))$ looks like:



Q: What are the directions of the unbounded rays in trop(V(f))? trop(V(g))? trop(V(h))?
(1,0), (0,1) and (−1,−1).

Q: What is the number of rays in each of these directions? (usually 2)

Q: What is deg(f)? deg(g)? deg(h)? All have deg 2

With appropriate notion of multiplicity:
(1) # of rays in each direction = degree of poly
(2) At each vertex of graph, the sum of vectors emanating from vertex add to zero.

Tropical curves in plane & triangulations of point configurations

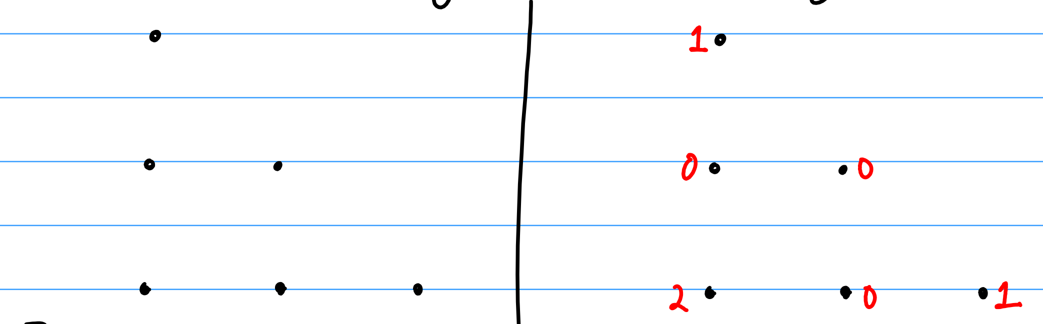
Choose $d \in \mathbb{Z}^+$. Let $A_d = \{(a,b) : a+b \le d, \text{ and } a,b \ge 0\}$.
Fix a polynomial
$$f = \sum_{(a,b)\in A_d} C_{ab}\, x^a y^b \quad \text{with} \quad C_{ab} \in \mathbb{C}\{\{t\}\}.$$

Consider the convex hull of all points $\{(a, b, \mathrm{val}(C_{ab})) : (a,b) \in A_d\}$ & take the (projection) of the set of <u>lower faces</u>.
(The faces you see if you look from $(0,0,-N)$ where $N \gg 0$)

This induces a <u>regular triangulation</u> $T(f)$ of A_d.

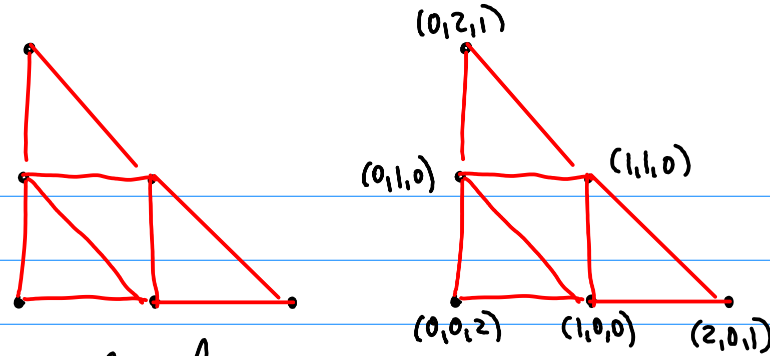
<u>Ex:</u> Let $d=2$. $A_2 = \{(0,0),(0,1),(1,0),(2,0),(1,1),(0,2)\}$
Let $f = tx^2 + 2xy + 3ty^2 + 5x + 7y - (t^2+t^5)$.

A_2:



points of A_2 | points of A_2 w/ each "height" written next to the point

Imagine lifting each point to corresp height, taking convex hull, then looking from below:

We will see: 

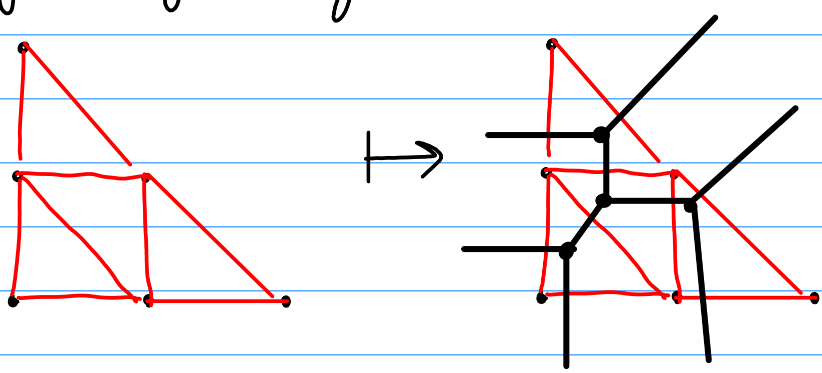
This is called a regular triangulation of A_2.

Def: The dual graph to a triangulation T has a vertex for every triangle, & an edge for every edge of T.

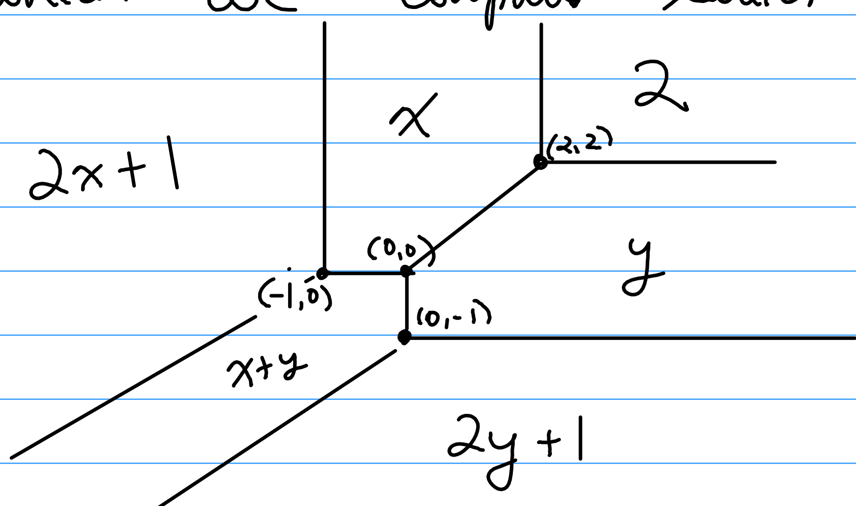
Two types of edges:

- finite edges joining two adjacent triangles, which are orthogonal to common edge
- infinite edges joining a triangle to the exterior, which are orthogonal to edge of triangle.

Ex:

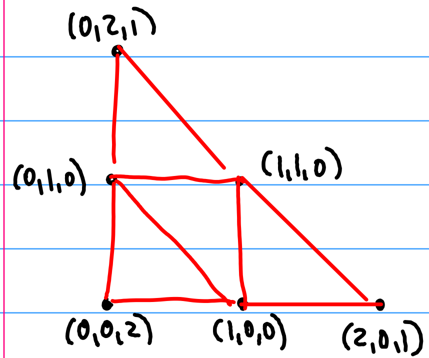


Compare this dual graph to $\operatorname{trop}(V(f))$ which we computed earlier:



Thm: Under the transformation $(x,y) \mapsto (-x,-y)$, $\mathrm{trop}(V(f))$ is a dual graph to the regular triangulation $T(f)$.

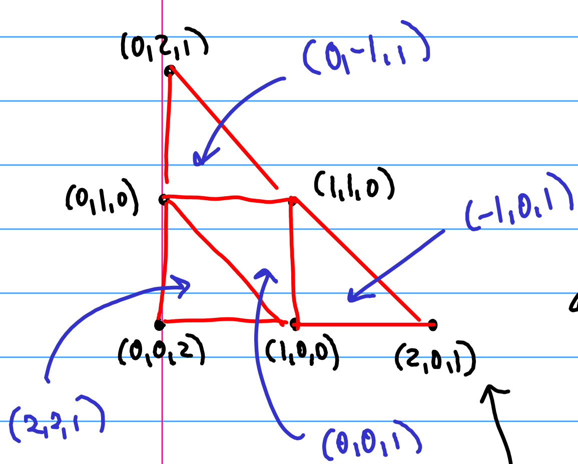
Pf idea for this example:



Since the triangles shown are lower faces of a convex polytope, for each face there is a linear function $g: \mathbb{R}^3 \to \mathbb{R}$ which is minimized at that face.

If $g(x,y,z) = ax+by+cz$, use (a,b,c) as shorthand for g.

Note: Can normalize s.t. $c=1$.



← Labels of faces show each g

Now compare w/ the tropical hypersurface!

Denote the vertex coordinates by $v_i \in \mathbb{R}^3$.
The equation that $g \cdot v_1 = g \cdot v_2 = g \cdot v_3 < g \cdot v_i$ for $i \neq 1,2,3$ is the same as the equation expressing what it means for 3 terms in $\min(2x+1,\ x+y,\ 2y+1,\ x,\ y,\ 2)$ to be tied as the minimums.

Exercises for Tropical Geometry

E (1) Compute the following quantities
(a) $(5 \oplus 6) \otimes 7$
(b) $2 \otimes (1 \oplus 3)$
(c) $-4 \otimes (-3, 1)$

E (2) Tropicalize the following polynomials
(a) $t^3 x + (t + 3t^2 + 5t^4) y + t^{-2}$
(b) $(t^{-1} + 1) x + (3t + t^3) xy + 5t^4$
(c) $t x^2 - 7y^2 + t^{-1} xy + tx + 1$

E (3) Check that $(\mathbb{R} \cup \infty, \oplus, \otimes)$ is a semiring

E (4) Let k be a field, and k^* the nonzero elements of k. A <u>valuation</u> on k is a function $v : k \to \mathbb{R} \cup \infty$ s.t.
(1) $v(a) = \infty$ iff $a = 0$
(2) $v(ab) = v(a) + v(b)$
(3) $v(a+b) \geq \min(v(a), v(b))$ for all $a, b \in k^*$

Check that the function $\mathrm{val} : K^* \to \mathbb{R}$ which we defined earlier for Puisseux series is a valuation.

M (5) Let $f \in K[x_1, \dots, x_n]$ and write f as

$$f = \sum_{u \in \mathbb{N}^n} c_u x^u \quad \text{where } x^u := \prod_{i=1}^{n} x_i^{u_i}.$$

Show that if $w \in \operatorname{trop}(V(f))$ then the minimum among the set $\left\{ \operatorname{val}(c_u) + \sum_{i=1}^{n} u_i x_i \right\}_{\substack{u \in \mathbb{N}^n \\ c_u \neq 0}}$ is achieved at least twice.

M (5) Draw a picture of the tropical curve corresponding to the following polynomials in $K[x,y]$:

(a) $f = (t^{-1}+1)x + (t^2 - 3t^3)y + 5t^4$

(b) $f = t^3x^2 + xy + ty^2 + tx + y + 1$

(c) $f = tx^2 + 4xy - 7y^2 + 8$

(d) $f = t^6x^3 + x^2y + xy^2 + t^6y^3 + t^3x^2 + t^{-1}xy + t^3y^2 + tx + ty + 1.$

H (6) Prove the following statement.

Thm: After applying transformation $(x,y) \mapsto (-x,-y)$, $\operatorname{trop}(V(f))$ is a dual graph to the regular triangulation $T(f)$.