Contents

1 Introduction

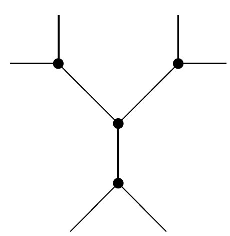
Recall that a combinatorial model is an explicit realization of the exchange graph, with combinatorial recipes for extracting data about each seed. Ideally, complete data, including cluster variables, but if not, other useful data. In this lecture, we're going to describe a *framework* for an exchange matrix B. Roughly, this will be an n-regular graph with edges labeled by vectors. A priori, we *don't know* that the graph has anything to do with the exchange graph. Properties of *the labeling tell you* that. The labeling will tell you principal coefficients (quite directly!), $\mathbf{g}$-vectors, exchange matrices, and, conjecturally, denominator vectors. The definitions and results in this paper are in a joint paper, in preparation, with David Speyer.

After this lecture, I'll talk about how to create frameworks from the combinatorics of *Coxeter groups*, and in particular *sortable elements*, *Cambrian lattices* and *Cambrian fans*. (This is also in the paper with Speyer.) Although the Cambrian stuff appears last, in fact, it was the Cambrian combinatorics that helped us figure out what the general definition of a framework should be.

2 Frameworks

Quasi-graphs

Quasi-graph: a hypergraph with edges of size 1 or 2. *Edges of size two* are edges in the usual graph-theoretic sense. *Edge of size* 1 are *half-edges*, dangling from a vertex. (Not to be confused with "self-edges" or "loops.") We will assume our quasi-graphs are *simple*, meaning no loops or multiple edges. Here is a quasi-graph with 4 vertices, 3 full edges and 6 half-edges.



The *degree* of a vertex is the total number of edges (including half-edges) incident to the vertex. The quasi-graph is *n-regular* if every vertex has degree n. Our example is 3-regular. A quasi-graph G is *connected* if the graph obtained from G by ignoring half-edges is connected in the usual sense. *Incident pairs* are (v, e) where v is a vertex contained in an edge e. Let $I(v)$ denote the set of edges e incident to a vertex v.

B defines a generalized Cartan matrix

We start with an exchange matrix B. This defines a generalized Cartan matrix A: Put 2's on the diagonal, and make all nonzero off-diagonal entries negative.

$$\begin{bmatrix} 0 & 0 & 3 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 2 & 0 & 1 \\ -3 & 0 & -1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & -3 & -1 \\ 0 & 2 & -1 & 0 \\ -1 & -2 & 2 & -1 \\ -3 & 0 & -1 & 2 \end{bmatrix}$$

So we have simple roots $\alpha_1, \dots, \alpha_n$ and simple co-roots $\alpha_1^\vee \dots, \alpha_n^\vee$, both bases for V. Also fundamental weights $\rho_1, \dots, \rho_n$, the basis of V^* dual to the simple co-roots. There is a symmetric bilinear form K on V such that $A = (a_{ij})$ with $a_{ij} = K(\alpha_i^\vee, \alpha_j)$. For each root β, there is a corresponding reflection t with $t(x) = x - K(\beta^\vee, x)\beta_i$.

Reflection frameworks

A *reflection framework* is a connected n-regular quasi-graph G equipped with a labeling C of each incident pair by a vector $C(v,e)$ in V, satisfying certain conditions. The set of labels on v is $C(v)$.

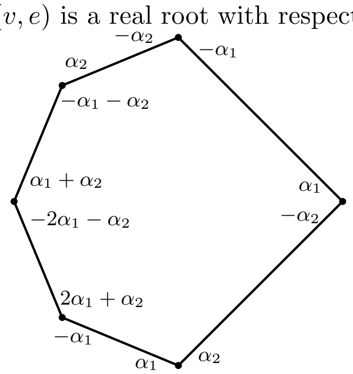
Base condition: There exists a vertex v_b with $C(v_b) = \Pi$.

Root condition: Each label $C(v,e)$ is a real root with respect to the Cartan matrix A.

Example:

$B = \left[\begin{smallmatrix} 0 & 2 \\ -1 & 0 \end{smallmatrix}\right]$

$A = \left[\begin{smallmatrix} 2 & -2 \\ -1 & 2 \end{smallmatrix}\right]$



Bilinear forms derived from B

The exchange matrix B defines a bilinear form ω by $\omega(\alpha_i^\vee, \alpha_j) = b_{ij}$. This form is the key to the definition of a framework.

Exercise 4Aa. *Show that ω is skew-symmetric.*

The *Euler form* E associated to B is $E(\alpha_i^\vee, \alpha_j) = \begin{cases} \min(b_{ij}, 0) & \text{if } i \neq j, \text{ or} \\ 1 & \text{if } i = j. \end{cases}$

Exercise 4Ab. *Show that $\omega(\beta, \gamma) = E(\beta, \gamma) - E(\gamma, \beta)$ and $K(\beta, \gamma) = E(\beta, \gamma) + E(\gamma, \beta)$.*

The Reflection condition

Reflection condition: Suppose v and v' are distinct vertices incident to the same edge e. Let $\beta = C(v,e)$ and let t be the reflection associated to β. Let β_t be the *positive* root associated to t, so $\beta = \pm\beta_t$. Recall that $t\gamma = \gamma - K(\beta^\vee, \gamma)\beta_i$. Let $\gamma \in C(v)$. Then $C(v')$ contains the root

$$\gamma' = \begin{cases} t\gamma & \text{if } \omega(\beta_t^\vee, \gamma) \geq 0, \text{ or} \\ \gamma & \text{if } \omega(\beta_t^\vee, \gamma) < 0. \end{cases}$$

Exercise 4Ac. *The Reflection condition implies that $-\beta \in C(v')$. If we reverse the roles of v and v' above and consider the root $\gamma' \in C(v')$, then the Reflection condition is the assertion that $\gamma \in C(v)$.*

The Euler conditions

Let v be a vertex of G. Let e and f be distinct edges incident to v. Write $\beta = C(v,e)$ and $\gamma = C(v,f)$. Then

(E0) At least one of $E(\beta,\gamma)$ and $E(\gamma,\beta)$ is zero.

(E1) If $\beta \in \Phi_+$ and $\gamma \in \Phi_-$ then $E(\beta,\gamma) = 0$.

(E2) If $\operatorname{sgn}(\beta) = \operatorname{sgn}(\gamma)$ then $E(\beta,\gamma) \leq 0$.

In light of the following facts:

$E(\alpha_i^\vee, \alpha_j) = \begin{cases} \min(b_{ij}, 0) & \text{if } i \neq j, \text{ or} \\ 1 & \text{if } i = j \end{cases}$

$\omega(\beta,\gamma) = E(\beta,\gamma) - E(\gamma,\beta)$, and

$K(\beta, \gamma) = E(\beta, \gamma) + E(\gamma, \beta)$,

we see that

(E0) implies $\omega(\beta, \gamma) = \pm K(\beta, \gamma)$.

(E1) implies $\omega(\beta, \gamma) = -K(\beta, \gamma)$ (conditionally).

(E2) implies $K(\beta, \gamma) \leq 0$ (conditionally).

3 Frameworks and exchange graphs

Reflection frameworks, frameworks and a result

A *(weak) reflection framework* is a pair (G, C) such that

- G is a connected n-regular quasi-graph, and
- C is a labeling of each incident pair by a vector $C(v, e)$ in V satisfying
  - the Base condition,
  - the Root condition,
  - the Reflection condition, and
  - the Euler conditions.

We'll allude later to the more general definition of a *framework*: a triple $(G, C, C^\vee)$ satisfying some conditions. Given a reflection framework, we can the *co-labeling* $C^\vee$ by $C^\vee(v, e) = (C(v, e))^\vee$. We'll state results in terms of frameworks.

Proposition 4A.1. *For any vertex in a framework, the vectors $C(v)$ are a $\mathbb{Z}$-basis for the root lattice $\mathbb{Z}\Pi$. In particular, $C(v)$ is a basis for V.*

The first main frameworks result

Theorem 4A.2. *Suppose $(G, C, C^\vee)$ is a framework for B and let $(\mathbf{x}, \mathbf{y}, B)$ be a labeled seed. Then there exists a covering map $v \mapsto \mathrm{Seed}(v) = (\mathbf{x}^v, \mathbf{y}^v, B^v)$ from G to part of the exchange graph $\mathrm{Ex}(\mathbf{x}, \mathbf{y}, B)$, such that the exchange matrix $B^v = [b^v_{ef}]_{e,f \in I(v)}$ has $b^v_{ef} = \omega(C^\vee(v, e), C(v, f))$.*

This statement is not quite true, but we'll fix it later. Why *part of* the exchange graph? *Half-edges.*

Example

$$b^v_{ef} = \omega(C^\vee(v, e), C(v, f))$$

$B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$

$\omega(\alpha_1^\vee, \alpha_2) = 2$

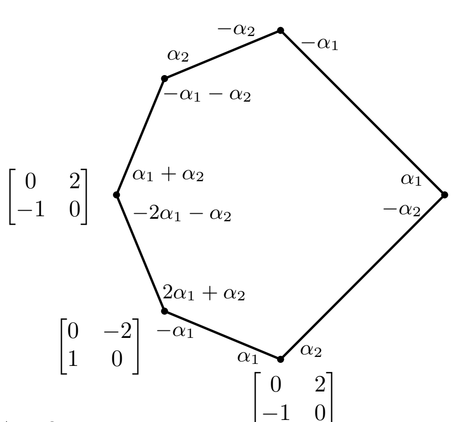
$\omega(\alpha_2^\vee, \alpha_1) = -1$

$(-2\alpha_1 - \alpha_2)^\vee = -\alpha_1^\vee - \alpha_2^\vee$

$(\alpha_1 + \alpha_2)^\vee = \alpha_1^\vee + 2\alpha_2^\vee$

$\omega(\alpha_1^\vee + 2\alpha_2^\vee, -2\alpha_1 - \alpha_2) = -2 + 4 = 2$

$\omega(-\alpha_1^\vee - \alpha_2^\vee, \alpha_1 + \alpha_2) = -2 + 1 = -1$



The other main frameworks result

Theorem 4A.3. *Suppose* $(G, C, C^\vee)$ *is a framework for* B *and let* $\mathcal{A}_\bullet(B)$ *be the cluster algebra with principal coefficients. Let* v *be a vertex of* G*. Then*

1. *In the bottom half of the extended exchange matrix* $\widetilde{B}^v$ *associated to* $\mathrm{Seed}(v)$*, the column indexed by* $e \in I(v)$ *is given by the coordinates of* $C(v, e)$ *in the basis* Π *of simple roots.*

2. *The rows of the bottom half of* $\widetilde{B}^v$ *are sign-coherent.*

3. *To compute the* $\mathbf{g}$*-vectors associated to* $\mathrm{Seed}(v)$*, find the dual basis to* $C^\vee(v)$*. Each* $\mathbf{g}$*-vector is given by the coordinates of a dual basis vector in terms of fundamental weights.*

4. *The* $\mathbf{g}$*-vectors associated to* $\mathrm{Seed}(v)$ *are a basis for the weight lattice.*

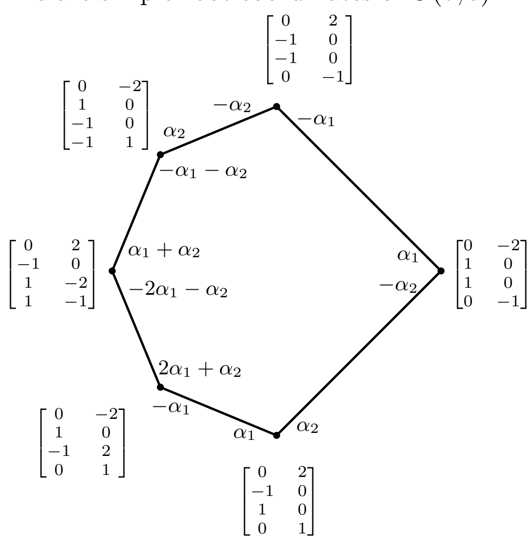
5. *The* F*-polynomial associated to* $\mathrm{Seed}(v)$ *has constant term* 1*.*

Example

Bottom half of $\widetilde{B}^v$: columns are simple-root coordinates of $C(v, e)$.

$$B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$$



In a general framework (i.e. not a reflection framework), we don't require that the labels are roots (the Root condition), so we can't define a co-labeling by just applying some "$^\vee$" operator to labels. (Recall why: Given a label γ, we don't know that $K(\gamma, \gamma) > 0$.) Instead, we define a labeling C and a co-labeling $C^\vee$ separately. The Base condition is expanded to require not only $C(v_b) = \Pi$, but also $C^\vee(v_b) = \Pi^\vee$. The Root condition is replaced by weaker requirements: Each label is either positive or negative (the Sign condition), and each co-label is a positive multiple of the corresponding co-label (the Co-label condition). The Reflection condition:

$$C(v') \text{ contains } \gamma' = \begin{cases} t\gamma & \text{if } \omega(\beta_t^\vee, \gamma) \geq 0, \text{ or} \\ \gamma & \text{if } \omega(\beta_t^\vee, \gamma) < 0. \end{cases}$$

is replaced by the Transition condition:

$$C(v') \text{ contains } \gamma' = \gamma + [\mathrm{sgn}(\beta)\omega(\beta^\vee, \gamma)]_+ \beta$$

and a similar Co-transition condition.

To get a sense of why reflection frameworks are a special case of frameworks: Recall (again) that $t\gamma = \gamma - K(\beta^\vee, \gamma)\beta_i$. Recall also that the Euler condition (E0) on a reflection framework implies that $\omega(\beta, \gamma) = \pm K(\beta, \gamma)$. The proof that frameworks recover extended exchange matrices is actually straightforward from the Transition and Co-transition conditions, once one thinks to try proving it.

Necessity of frameworks

Frameworks are a way of making models for cluster algebras. In fact, in some sense they are the *only* way. Assuming some conjectures from (CDM) and (CA IV), every exchange matrix admits a framework that models the entire cluster pattern. So, any model for the exchange graph that gives g-vectors or principal coefficients contains information equivalent in a framework.

4 Complete, exact, polyhedral, and well-connected frameworks

Complete frameworks

The simplest additional condition one can put on a framework is to require that there be no half-edges. Thus the framework is built on an n-regular *graph* G, and one can show that this implies that the map $v \mapsto \mathrm{Seed}(v)$ is surjective from G to the exchange graph. This observation, together with Theorem 4A.3 (the second main theorem on frameworks) imply the following:

Theorem 4A.4. *If a complete framework exists for B, then Conjectures 2.12, 2.14, and 2.17 all hold for B. If in addition a complete framework exists for $-B$, then Conjecture 2.13 also holds for B.*

2.12: Each F-polynomial has constant term 1.

2.13: Each F-polynomial has a unique max.-degree monomial.

2.14: For each cluster, the $\mathbf{g}$-vectors are a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.

2.17: The rows of the bottom half of principal-coeff extended exchange matrices are sign-coherent.

Ample frameworks (or, "Why Theorem 4A.2 is not quite true.")

Theorem 4A.2 says: for a framework $(G, C, C^\vee)$, there's a covering map $v \mapsto \mathrm{Seed}(v) = (\mathbf{x}^v, \mathbf{y}^v, B^v)$ from G to part of $\mathrm{Ex}(\mathbf{x}, \mathbf{y}, B)$. Really, *a priori*, we don't know that the following type of thing doesn't happen:

Recall our examples with $B = \left[\begin{smallmatrix} 0 & 2 \\ -1 & 0 \end{smallmatrix}\right]$. Then $\mathrm{Ex}(\mathbf{x}, \mathbf{y}, B)$ is a 6-cycle. We can imagine having a framework $(G, C, C^\vee)$ with G a 3-cycle, that would still encodes the combinatorics of exchange matrices, principal coefficients, and $\mathbf{g}$-vectors. One would traverse G twice to traverse $\mathrm{Ex}(\mathbf{x}, \mathbf{y}, B)$ once.

The correct statement of Theorem 4A.2 asserts a covering map from the *universal cover* of G to $\mathrm{Ex}(\mathbf{x}, \mathbf{y}, B)$. If Theorem 4A.2 is true for $(G, C, C^\vee)$ as we stated it earlier (in the principal coefficients case), then $(G, C, C^\vee)$ is an *ample framework*. Conjecture 2.15 ("Different cluster monomials have different $\mathbf{g}$-vectors") would imply that non-ample frameworks can't exist.

Exact frameworks

A framework is *injective* if, for every pair u, v of vertices of G, the following three conditions are equivalent:

$$C(u) = C(v); \quad C^\vee(u) = C^\vee(v); \quad u = v.$$

A framework is *exact* if it is injective and ample.

Theorem 4A.5. *If $(G, C, C^\vee)$ is a complete, exact framework. Then the map $v \mapsto \mathrm{Seed}(v) = (\widetilde{B}^v, X^v)$ is a graph isomorphism from G to the principal-coefficients exchange graph $\mathrm{Ex}_\bullet(B)$.*

Ampleness is a condition whose definition relies on *a priori* knowledge of the principal coefficients exchange graph. But, there is a combinatorial "simple connectedness" condition that implies ampleness. Thus, we can satisfy the hypotheses of Theorem 4A.5 with no *a priori* knowledge of the cluster pattern.

Corollary 4A.6. *If a complete, exact framework exists for B, then Conjecture 2.16 holds for B.*

2.16: In the principal-coefficients case, if seeds have equivalent extended exchange matrices, then the seeds are equivalent.

Polyhedral frameworks

Proposition 4A.7. *For any vertex v in a framework $(G, C, C^\vee)$, the labels $C(v)$ are a $\mathbb{Z}$-basis for the root lattice, and the co-labels $C^\vee(v)$ are a $\mathbb{Z}$-basis for the weight lattice.*

In particular, $C(v)$ is a linearly independent set.

Exercise 4Ad. *Prove linear independence of $C(v)$ in the case of a reflection framework.*

Define $\text{Cone}(v)$ to be the full-dimensional, simplicial cone

$$\bigcap_{e \in I(v)} \{x \in V^* : \langle x, C^\vee(v,e) \rangle \geq 0\}.$$

The framework $(G, C, C^\vee)$ is *polyhedral* if the cones $\text{Cone}(v)$ form a *fan*, with each vertex v labeling a distinct maximal cone.

Simplicial cones

(Simplicial: just for convenience in explaining.)

A *simplicial cone* U in V^* is:

- the nonnegative linear span of a basis for V^*.

$$U = \mathbb{R}_{\geq 0} v_1 + \cdots \mathbb{R}_{\geq 0} v_n.$$

Equivalently:

- the intersection of n "independent" halfspaces.

$$U = \bigcap_{i=i,\ldots,n} \{x \in V^* : \langle x, \beta_i \rangle \geq 0\},$$

  where $\beta_1, \ldots, \beta_n$ are a basis for V.

The basis $v_1, \ldots, v_n$ is the dual basis to $\beta_1, \ldots, \beta_n$ (up to positive scaling).

Simplicial fans

A *face* of a simplicial cone U is

- the nonnegative linear span of a subset of the vectors defining U

Equivalently:

- the intersection U with some collection of its defining hyperplanes.

A *(full-dimensional) simplicial fan* is a collection of simplicial cones, and their faces such that, for each pair of cones in the collection, their intersection is a face of each. (Usually, the fan is the collection consisting of the n-dimensional cones, together with all their faces.)

Back to polyhedral frameworks

Recall parts of Theorem 4A.3:

1. In the bottom half of the extended exchange matrix $\widetilde{B}^v$ associated to $\text{Seed}(v)$, the column indexed by $e \in I(v)$ is given by the coordinates of $C(v,e)$ in the basis Π of simple roots.

3. To compute the **g**-vectors associated to $\text{Seed}(v)$, find the dual basis to $C^\vee(v)$. Each **g**-vector is given by the coordinates of a dual basis vector in terms of fundamental weights.

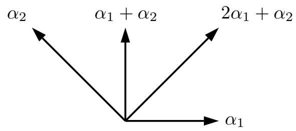
So, **g**-vectors are span rays in this fan, and principal coefficients define "walls." (C.f. (NZ).)

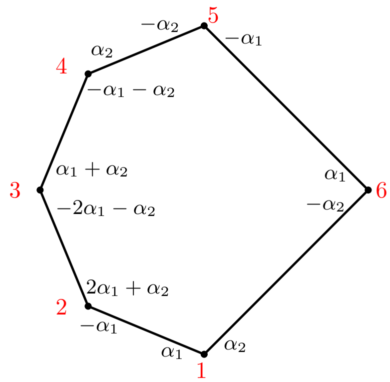
Example

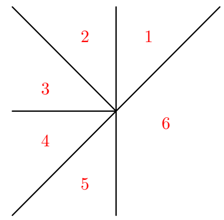
$$\mathrm{Cone}(v) = \bigcap_{e \in I(v)} \{x \in V^* : \langle x, C^\vee(v,e)\rangle \geq 0\}.$$

Example:
$B = \left[\begin{smallmatrix} 0 & 2 \\ -1 & 0 \end{smallmatrix}\right]$

$A = \left[\begin{smallmatrix} 2 & -2 \\ -1 & 2 \end{smallmatrix}\right]$







Well-connected polyhedral frameworks

A polyhedral framework is *well-connected* if it has the following property: If F is a face of $\mathrm{Cone}(v)$ and of $\mathrm{Cone}(v')$, then there exists a path $v = v_0, v_1, \ldots, v_k = v'$ in G such that F is a face of v_i for all i from 0 to k.

Theorem 4A.8. *If $(G, C, C^\vee)$ is a well-connected polyhedral framework, then* **g***-vector cones for seeds in the image of $v \mapsto \mathrm{Seed}(v)$ form a fan, identical to the fan defined by $(G, C, C^\vee)$.*

Corollary 4A.9. *If $(G, C, C^\vee)$ is a complete, well-connected polyhedral framework for B, then* **g***-vector cones for seeds in the cluster pattern form a fan, identical to the fan defined by $(G, C, C^\vee)$.*

Corollary 4A.10. *If a complete, exact, well-connected polyhedral framework exists for B, then Conjecture 2.15 holds for B.*

2.15: Different cluster monomials have different **g**-vectors.

Summary

- If you can build a complete, exact, well-connected polyhedral framework for B, then you have proved a number of structural conjectures for B. (In fact, more than we have mentioned.)
- Checking that you have built one requires *no a priori knowledge of the cluster pattern*. Just combinatorial conditions on the labels. (Ampleness comes via combinatorial simple-connectedness.)
- An easily overlooked feature of the setup: The roots/co-roots/weights setup allows us to handle the skew-symmetrizable (not-necessarily-skew-symmetric) case without trouble.

Constructing frameworks

- For the rest of the week, we'll discuss a construction of complete, exact, well-connected polyhedral frameworks in the case where B defines a Cartan matrix A of finite or affine type.
- David Speyer and Hugh Thomas have constructed complete, injective frameworks for skew-symmetric, acyclic B. These are exact, because Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky (QP2) have proved Conjectures 2.15 for skew-symmetric B, but as far as I know, Speyer and Thomas do not have an independent proof of ampleness.
- From Sergey Fomin and Dylan Thurston's paper (CATS II), we see how to construct complete, exact frameworks for any B fitting into the triangulated surfaces setup. (You'll learn about this setup next week from Gregg Musiker.) Is there low-hanging fruit there? Or do the triangulated-surfaces people already have everything we could get from frameworks?

References

(QP2) H. Derksen, J. Weyman, and A. Zelevinsky, “Quivers with potentials and their representations II: applications to cluster algebras.” Journal AMS **23**.

(CATS II) S. Fomin and D. Thurston, “Cluster algebras and triangulated surfaces II: Lambda lengths.” Preprint, 2008.

(CDM) S. Fomin and A. Zelevinsky, “Cluster algebras: Notes for the CDM-03 conference.” CDM 2003: Current Developments in Mathematics, International Press, 2004.

(CA IV) S. Fomin and A. Zelevinsky, “Cluster algebras IV: Coefficients.” Compositio Mathematica **143**.

(NZ) T. Nakanishi and A. Zelevinsky, “On tropical dualities in cluster algebras.” arXiv:1101.3736.

(FRM) N. Reading and D. Speyer, “Combinatorial frameworks for cluster algebras.” In preparation.

Exercises, in order of priority

There are more exercises than you can be expected to complete in a *half* day. Please work on them in the order listed. Exercises on the first line constitute a minimum goal. It would be profitable to work all of the exercises eventually.

4Aa, 4Ab,

4Ac, 4Ad.