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MSRI School #5: KP equation + total positivity ...

The KP equation is:

$$\frac{\partial}{\partial x}\left(-4\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)+3\frac{\partial^2 u}{\partial y^2}=0$$

- Introduced by Kadomtsev & Petviashvili in 1970 to study stability of certain solutions to KdV equation
- Further developed by Kyoto school: Sato, Hirota, Miwa, Jimbo, Date, ...
- Can be used to describe shallow water wave phenomena

First goal: Explain how to use point $A \in Gr_{k,n}(\mathbb{R})$ to construct solution to KP equation.

Fix $1 \le k < n$ and generic parameters $\kappa_1 < \dots < \kappa_n$.

Let $\{E_j : 1 \le j \le n\}$ be a set of exponential functions in $(t_1, \dots, t_n) \in \mathbb{R}^n$:

(*) $$E_j(t_1,\dots,t_n) := \exp\left(\sum_{i=1}^{n} \kappa_j^i t_i\right)$$

For $J \in \binom{[n]}{k}$, define $E_J := E_{j_1} E_{j_2} \dots E_{j_k} \prod_{\ell < m} (\kappa_{j_m} - \kappa_{j_\ell})$

Let A be a full rank $k \times n$ real matrix.

Define $\tau_A(t_1,\dots,t_n) = \sum_{I \in \binom{[n]}{k}} \Delta_I(A)\, E_I$

Set $t_1 = x$, $t_2 = y$, $t_3 = t$, and treat the other t_i's as constants.

Theorem (Freeman–Nimmo '83):
$u_A(x,y,t) := 2 \frac{\partial^2}{\partial x^2} \ln \tau_A(x,y,t)$ is a solution to the KP equation.

Key idea of pf: Write $\tau_A(x,y,t)$ as a Wronskian $Wr(f_1,\dots,f_k)$ where $(f_1,\dots,f_k)^t = A \cdot (E_1,\dots,E_n)^t$. (then computation ~ 2 pages)

Obs: If $\Delta_I(A) \geq 0 \ \forall I$, (i.e. $A \in (Gr_{k,n})_{\geq 0}$) then $u_A(x,y,t)$ is regular everywhere.

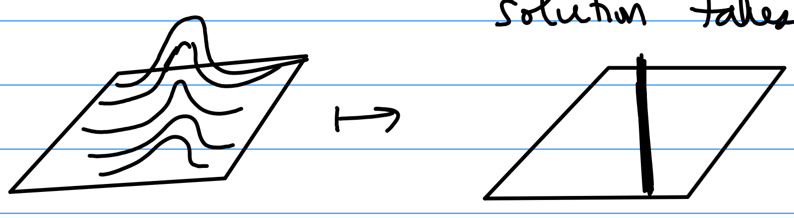
Goal: Study these solutions to the equation.

References: Kodama, Chakravarty, ...

End of this lecture based on joint work w/ Kodama "KP solitons, total positivity, & cluster algebras"

How can we visualize solutions?

Fix time t. Plot the points where the solution takes on local max



contour plot

Ex: Consider $A = (1 \ a)$ with $a > 0$. Choose $k_1 < k_2$

Then $\tau_A(x,y,t) = \Delta_1 E_1 + \Delta_2 E_2 = E_1 + aE_2$.

Write $E_1 = e^{\theta_1}$ and $aE_2 = e^{\theta_2}$. $(\theta_1 = k_1 x + k_1^2 y + k_1^3 t$ and $\theta_2 = \ln a + k_2 x + k_2^2 y + k_2^3 t)$

So $\tau_A(x,y,t) = e^{\theta_1} + e^{\theta_2} = 2e^{\frac{1}{2}(\theta_1+\theta_2)} \cosh \frac{1}{2}(\theta_1 - \theta_2)$

(Recall $\cosh x = \dfrac{e^x + e^{-x}}{2}$)

Computation $\Rightarrow u_A(x,y,t) = \frac{1}{2}(K_1-K_2)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_1-\theta_2)$

$$\left(\sinh x = \frac{1}{\cosh x} \right)$$

This is called a line-soliton solution
(maintains shape while traveling at constant speed)

Note: $\cosh x$ takes its min value at $x=0$ so $\sinh x$ is maximized at $x=0$.

$\therefore u_A(x,y,t)$ is maximized when $\theta_1=\theta_2$, ie. when

$$K_1 x + K_1^2 y + K_1^3 t = \ln a + K_2 x + K_2^2 y + K_2^3 t$$

$$\Leftrightarrow (K_1-K_2)x + (K_1^2-K_2^2)y + (K_1^3-K_2^3)t = \ln a$$

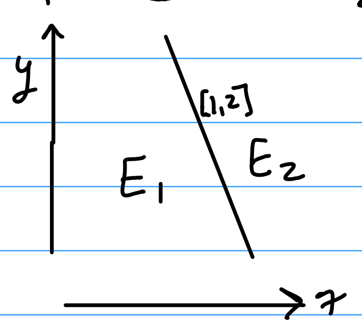
$$\Leftrightarrow x + (K_1+K_2)y + (K_1^2+K_1K_2+K_2^2)t = \frac{1}{K_1-K_2}\ln a$$

This is a line. Slope is determined by K_1+K_2.
We say the slope "is" K_1+K_2.
(This is proportional to the angle measured counterclock from pos. y-axis)

Since line is given by $\theta_1=\theta_2$, both exponential terms E_1 and aE_2 contribute equally to $T_A(x,y,t)$.

Since $K_1<K_2$,
when $x \ll 0$, $\theta_1>\theta_2$ so E_1 is the dominant exponential,
& when $x \gg 0$, $\theta_1<\theta_2$ so aE_2 is dominant.

Encode data by



Next: approximate contour plots by tropical curves

At "most" points (x,y,t),

$$T_A(x,y,t) = \sum_{I \in \binom{[n]}{k}} \Delta_I(A)\, E_I$$ will be dominated by one term.

Claim: In that case, if $T_A(x,y,t) \approx \Delta_J(A) E_J$ (some J) then $u_A(x,y,t) \approx 0$.

Pf: Recall $E_j(x,y,t) = \exp(\kappa_j x + \kappa_j^2 y + \kappa_j^3 t + \text{const})$

$E_J = \text{const} \cdot E_{j_1} E_{j_2} \cdots E_{j_k}$.

So
$$u_A(x,y,t) = 2 \frac{\partial^2}{\partial x^2} \ln T_A(x,y,t)$$
$$\approx 2 \frac{\partial^2}{\partial x^2} \ln (\Delta_J(A) E_J)$$
$$= 2 \frac{\partial^2}{\partial x^2} \left(\ln \Delta_J(A) + \ln(\text{const}) + \sum_{h=1}^{k} \ln(E_{j_h}) \right)$$
$$= 2 \frac{\partial^2}{\partial x^2} \left(\text{consts} + \sum_{h=1}^{k} (\kappa_j x + \text{stuff not depending on } x) \right)$$
$$= 0. \qquad \left(\text{since } \frac{\partial}{\partial x}\left(\sum_{h=1}^{k} \kappa_j \right) = 0\right)$$

$\therefore$ $u_A(x,y,t)$ has local max when two or more terms are tied.

To study this, set

$$\hat{f}_A(x,y,t) = \max \{ \Delta_J(A)\, E_J \}_{J \in \binom{[n]}{k}}$$

$$= \max\left\{ \Delta_J(A)\, E_{j_1} \cdots E_{j_k} \prod_{l<m} (\kappa_{j_m} - \kappa_{j_l}) \right\}_{J \in \binom{[n]}{k}}$$

$$= \max\left\{ \exp\left(\ln\left(\Delta_J(A) \prod (\kappa_{j_m} - \kappa_{j_l})\right) + \sum_{i=1}^{k} \left(\kappa_{j_i} x + \kappa_{j_i}^2 y + \kappa_{j_i}^3 t + \ldots\right) \right) \right\}_{J \in \binom{[n]}{k} \text{ and } \Delta_J(A) \neq 0}$$

Define $f_A(x,y,t)$ by

$$\max\left\{ \ln\left(\Delta_J(A) \prod (\kappa_{j_m} - \kappa_{j_l})\right) + \sum_{i=1}^{k} \left(\kappa_{j_i} x + \kappa_{j_i}^2 y + \kappa_{j_i}^3 t + \ldots\right) \right\}_{J \in \binom{[n]}{k} \text{ and } \Delta_J(A) \neq 0}$$

<u>Note:</u> A term dominates $\hat{f}_A(x,y,t)$ iff corresp term dominates $f_A(x,y,t)$.

<u>Def:</u> Given a solution $u_A(x,y,t)$ of KP eqn coming from τ_A, we define its <u>contour plot</u> $\mathcal{C}(u_A)$ to be the locus in $\mathbb{R}^3$ where $f_A(x,y,t)$ is not linear. [ie. where 2 or more exponentials dominate τ_A].
If we fix $t = t_0$, also define $\mathcal{C}_{t_0}(u_A)$ to be locus in $\mathbb{R}^2$ where $f_A(x,y,t=t_0)$ is not linear. (and call this <u>contour plot</u> also)

<u>Note:</u> It approximates the location of wave crests

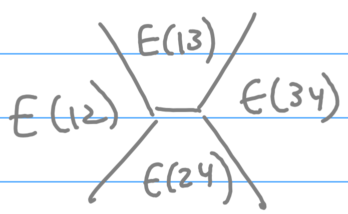
<u>Remark:</u> If each $\kappa_i \in \mathbb{Z}$, then $\mathcal{C}(u_A)$ is tropical hypersurface in $\mathbb{R}^3$ and $\mathcal{C}_{t_0}(u_A)$ is tropical curve in $\mathbb{R}^2$.

We'll focus on $\mathcal{C}_{t_0}(u_A)$'s.

From def, $\mathcal{C}_{t_0}(u_A)$ is piecewise linear. Each region of the complement is a domain of linearity for $f_A(x,y,t)$, associated w/ a unique <u>dominant</u> <u>exponential</u> $\Delta_J(A)\,E_J$.

<u>Lemma</u>: The index sets of the dominant exponentials of the τ-function in adj. regions of the contour plot in the x-y plane are of the form $I=\{i, m_2, \dots, m_k\}$ and $J=\{j, m_2, \dots, m_k\}$.

<u>N.B</u>: ↑ Under assumption that any phase shifts are negligable (ok if each $\frac{1}{k_{i+1}-k_i}$ of order 1)
Sometimes we might see a configuration like



But the length of this segment is very small

<u>Def</u>: We call the line separating the 2 adj regions in the Lemma a <u>line-soliton</u> of type $[i,j]$.

Convention: we put i and j in order with $i<j$

On this line,

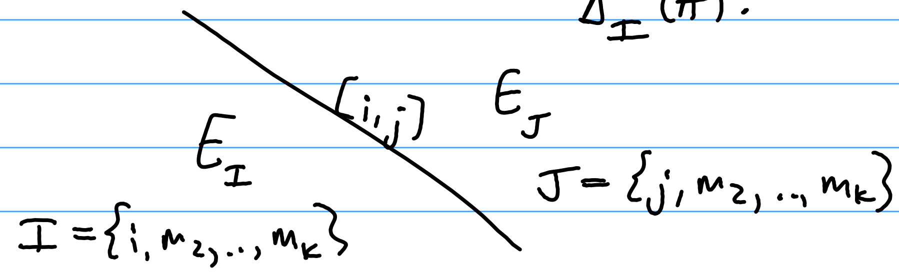
$$\Delta_I(A)\,E_I = \Delta_J(A)\,E_J,$$

$$\Rightarrow \Delta_I(A)\,K_I\,E_i \cancel{\prod_{l=2}^{k} E_{m_l}} = \Delta_J(A)\,K_J\,E_j \cancel{\prod_{l=2}^{k} E_{m_l}},$$

So the eqn for the line-soliton (of <u>type</u> $[i,j]$) is

$$x + (k_i + k_j)y + (k_i^2 + k_i k_j + k_j^2)t + \text{const} = -\frac{1}{k_j - k_i} \ln \frac{\Delta_J(A) k_J}{\Delta_I(A) k_I}.$$

Note: The slope "is" $k_i + k_j$, and location is determined by $\dfrac{\Delta_J(A)}{\Delta_I(A)}$.



Recall: $(Gr_{kn})_{\geq 0}$ has a decomposition into positroid cells $S_M^{tnn} = \{A \in (Gr_{kn})_{\geq 0} : \Delta_I(A) > 0 \text{ iff } I \in M\}$.

Positroid cells in $(Gr_{kn})_{\geq 0}$ in bijection with:

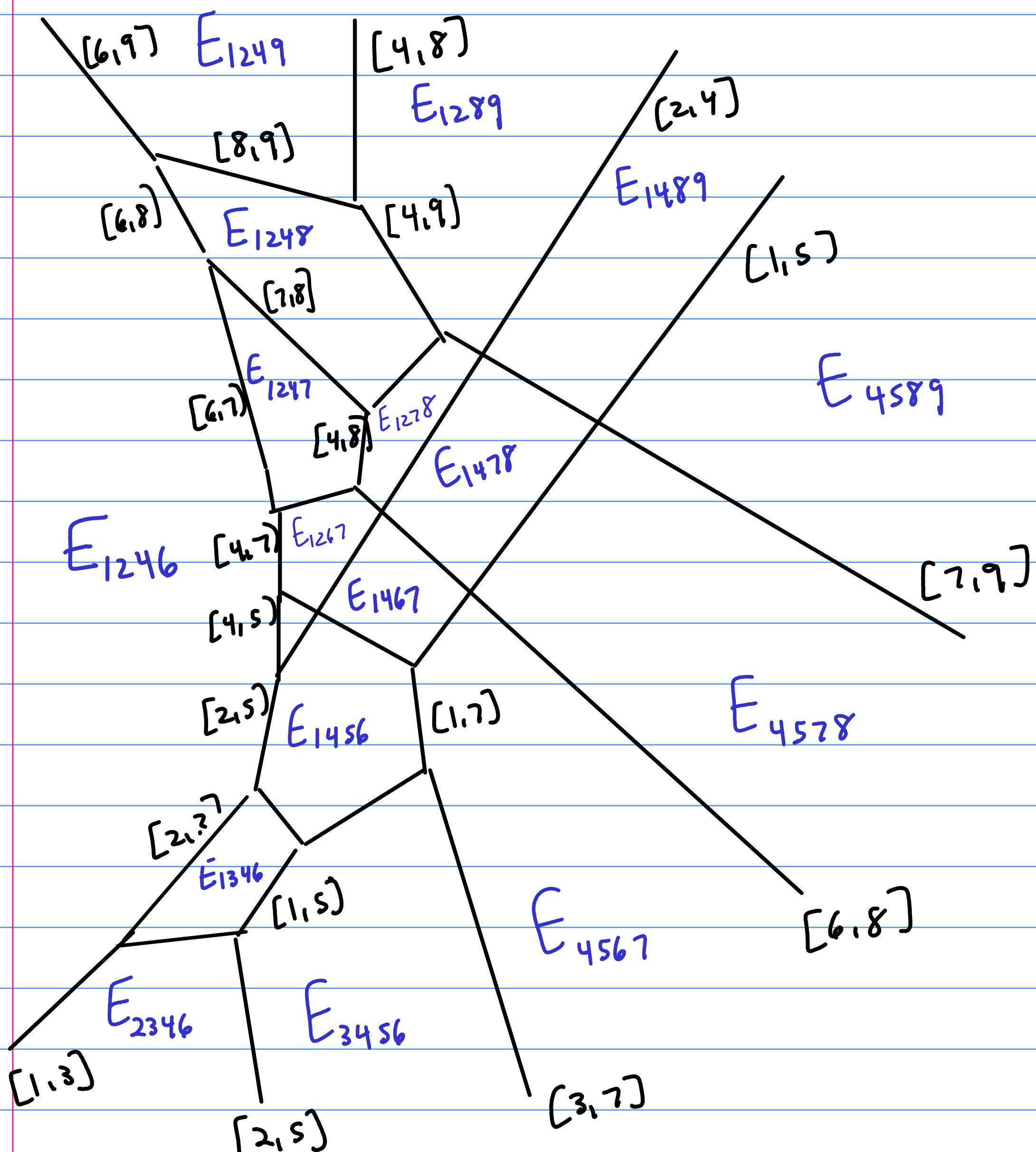
- ⅃-diagrams L in a $k \times (n-k)$ rectangle
- Dec perms π on n letters w/ k weak exc.

∴ we can denote positroid cells by S_L^{tnn} or S_π^{tnn}

Question: If $A \in S_M^{tnn}$, what can we say about the contour plots $\mathcal{C}_t(u_A)$?

Ex: A contour plot for an element $A \in S_\pi^{tnn}$
where $\pi = (5,4,1,8,2,9,3,6,7)$ ie.

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 4 & 1 & 8 & 2 & 9 & 3 & 6 & 7 \end{array}$$



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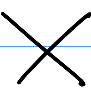
Q: Look at unbounded line-solitons. Observations?

Q: Remind you of plabic graphs?

Thm (Biondini-Chakravarty, C-Kodama, K-W.)
Let $A \in S_\pi^{tnn}$ where π is a derangement.
Then for any time t, the contour plot $\mathcal{C}_t(u_A)$ has precisely
k line-solitons at $y \gg 0$, labeled by exceedances of π
$n-k$ " " at $y \ll 0$, " " nonexceedances of π.

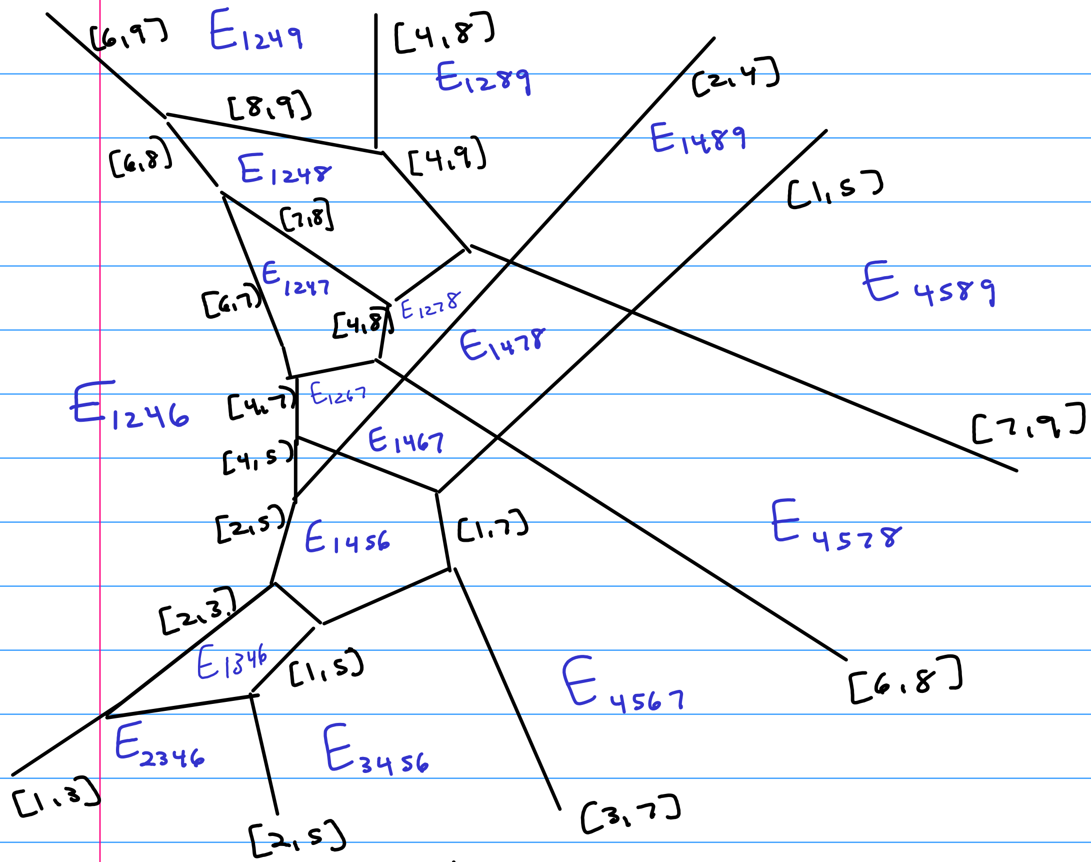
Now we want to relate these contour plots to plabic graphs.
Generically: lines meet at trivalent vertices or X-crossings



Algorithm:

1. Embed contour plot $\mathcal{C}_t(u_A)$ in disk, so that the unbounded solitons end at bdy vertices.
2. Associate color to each trivalent vertex:
   - ○ if there is a unique edge extending up
   - ● if there is a unique edge extending down
3. Recall that each unbounded soliton at $y \gg 0$ (resp $y \ll 0$) is labeled by $[i, \pi(i)]$ (resp $[\pi(i), i]$).
   Label the corresponding bdy vertex by $\pi(i)$.
4. Erase all edge & region labels of the contour plot.

This defines a generalized plabic graph which we call a soliton graph $G_t(u_A)$.


[6,9]
E1249
[4,8]
E1289
[2,4]
[8,9]
E1489
[6,8]
E1248
[4,9]
[1,5]
[7,8]
E1247
E4589
[6,7]
[4,8]
E1278
E1478
E1246
[4,7]
E1267
[7,9]
[4,5]
E1467
[2,5]
E1456
[1,7]
E4578
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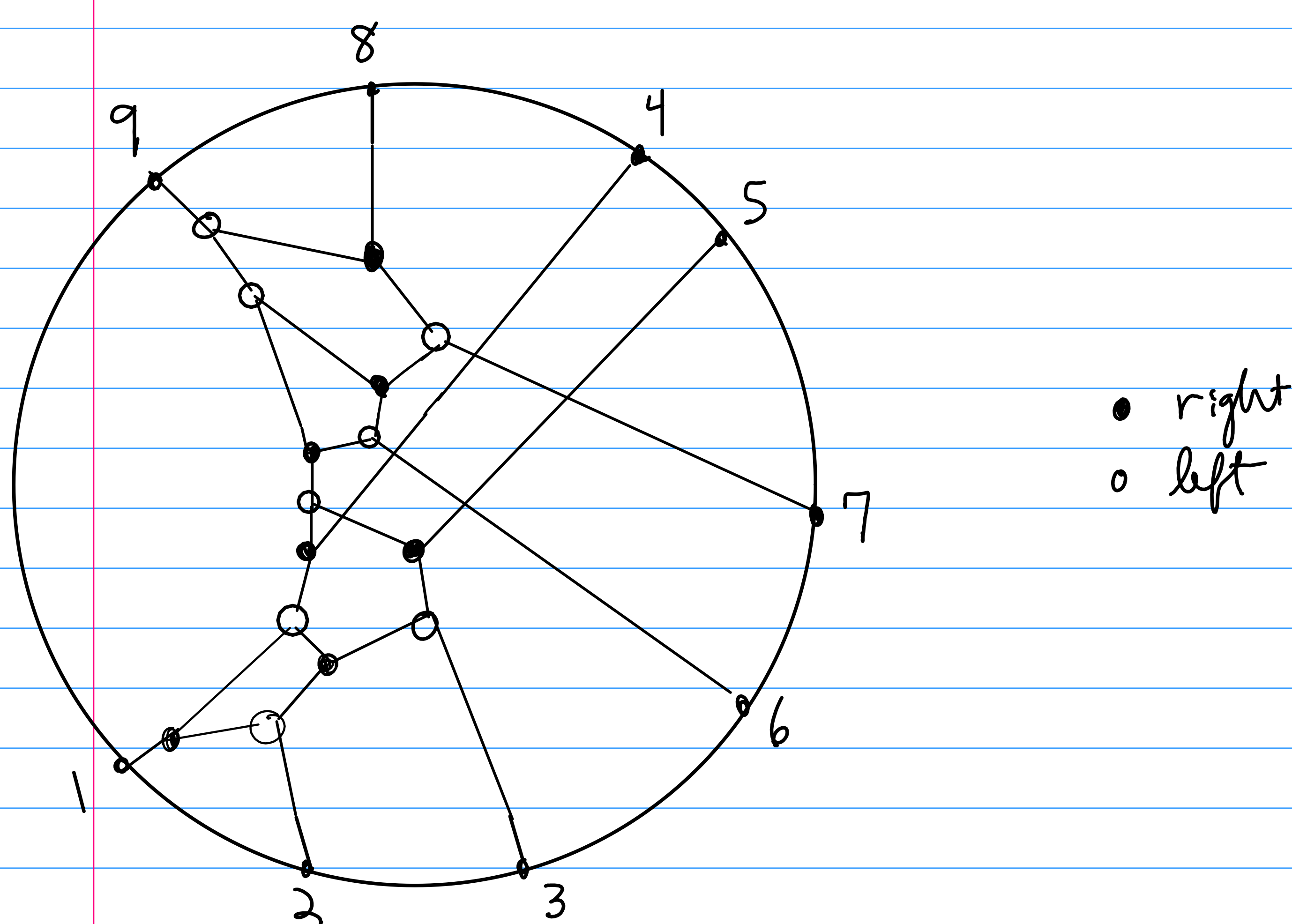
8
9
4
5
7
6
1
2
3
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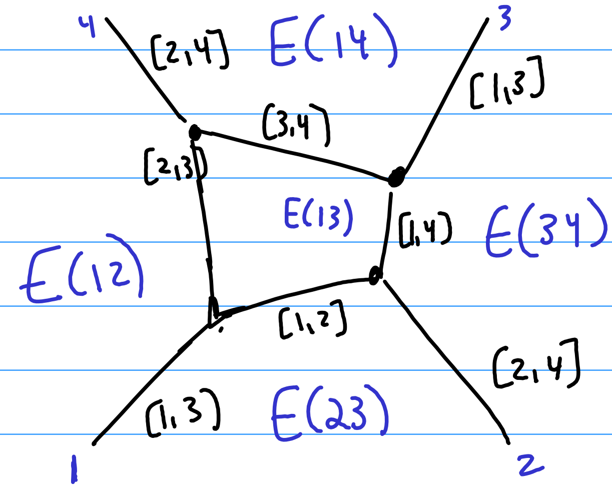
o left

Thm (Kodama-W.) If we use this algorithm & follow the "rules of the road" to label regions + edges, we will recover the original labeled contour plot.

Note: we stipulate that a trip goes straight through an X-crossing.

Ex: For $(Gr_{2,4})_{>0}$, we may get a contour plot that looks like this:



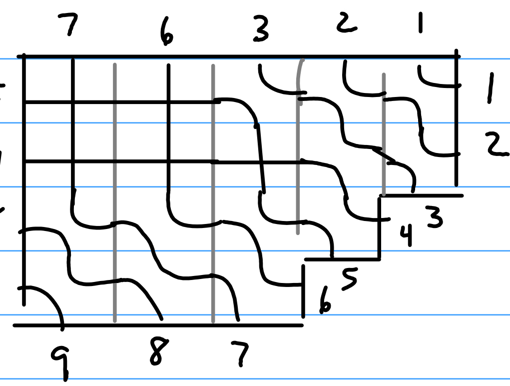
Recognize ??

Thm (KW) Suppose $A \in (Gr_{kn})_{>0}$, & that $G_t(u_A)$ is generic (no vertices of degree > 3). Then the soliton graph $G_t(u_A)$ is a reduced plabic graph. In particular, the label set for the regions of $G_t(u_A)$ gives a positivity test (cluster) for $(Gr_{kn})_{>0}$.

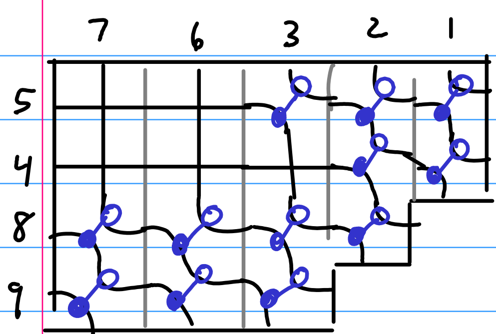
Thm (K-W): Let L be a Le-diagram & choose $A \in S_L^{tnn}$. The following procedure constructs the soliton graph $G_t(u_A)$ for t sufficiently small ($t \ll 0$).

0	0	+	+	+
0	0	0	+	+
+	+	+	+	
+	+	+		

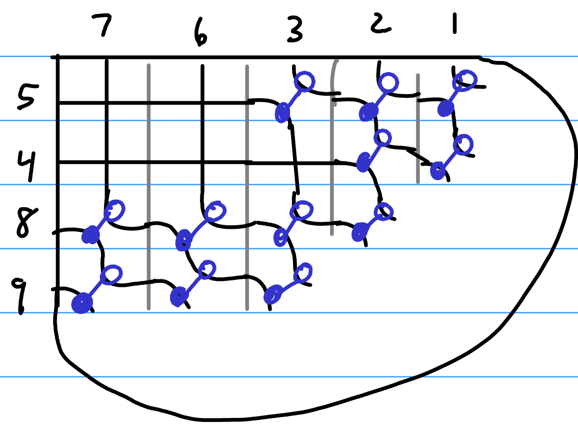
$\mapsto$



$\mapsto$



$\mapsto$

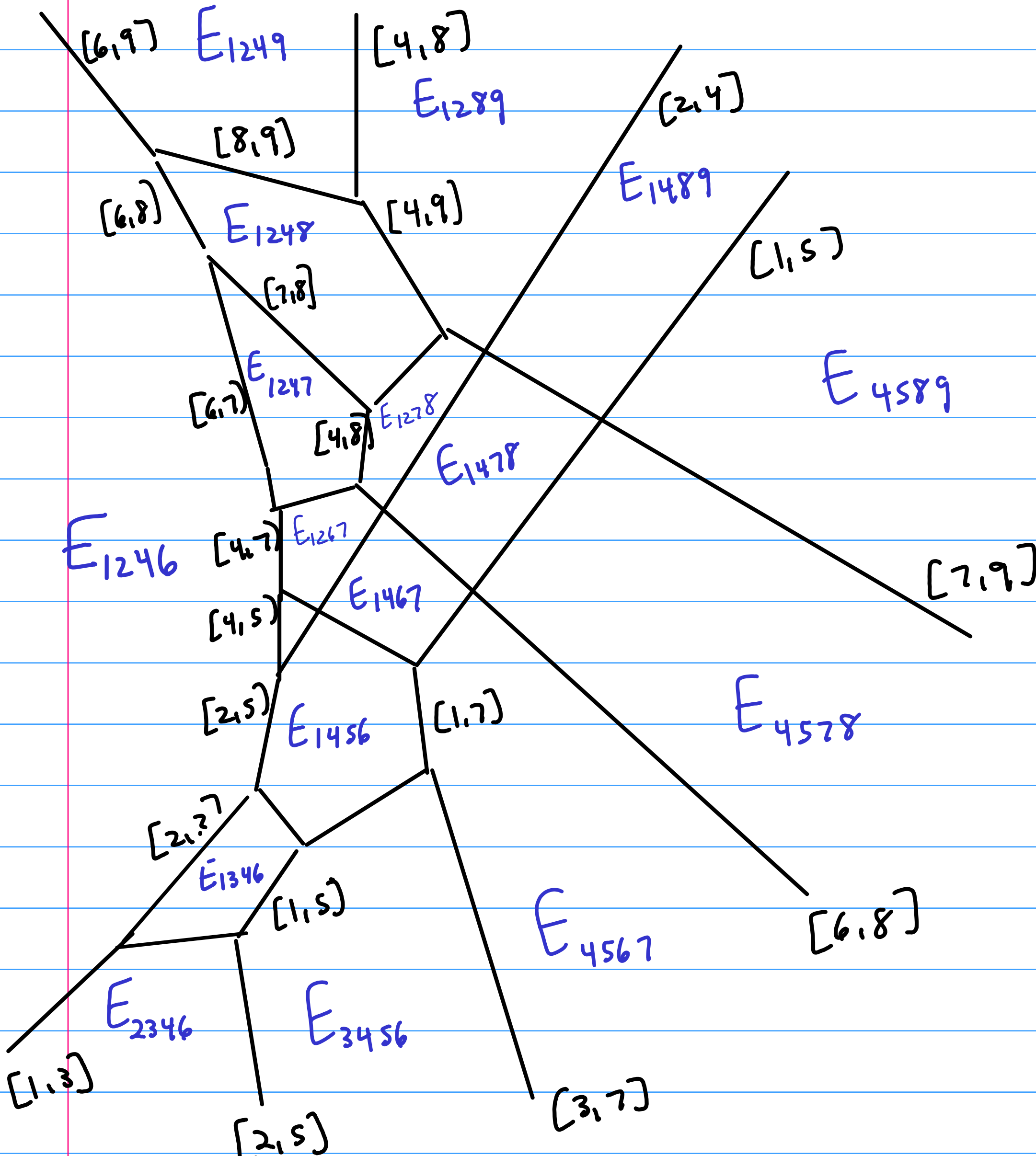


Check: After rotating, this is exactly the soliton graph we had before! (page 10)

Handout

Ex: A contour plot for an element $A \in S_\pi^{tnn}$ where $\pi = (5,4,1,8,2,9,3,6,7)$ ie.

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 4 & 1 & 8 & 2 & 9 & 3 & 6 & 7 \end{array}$$



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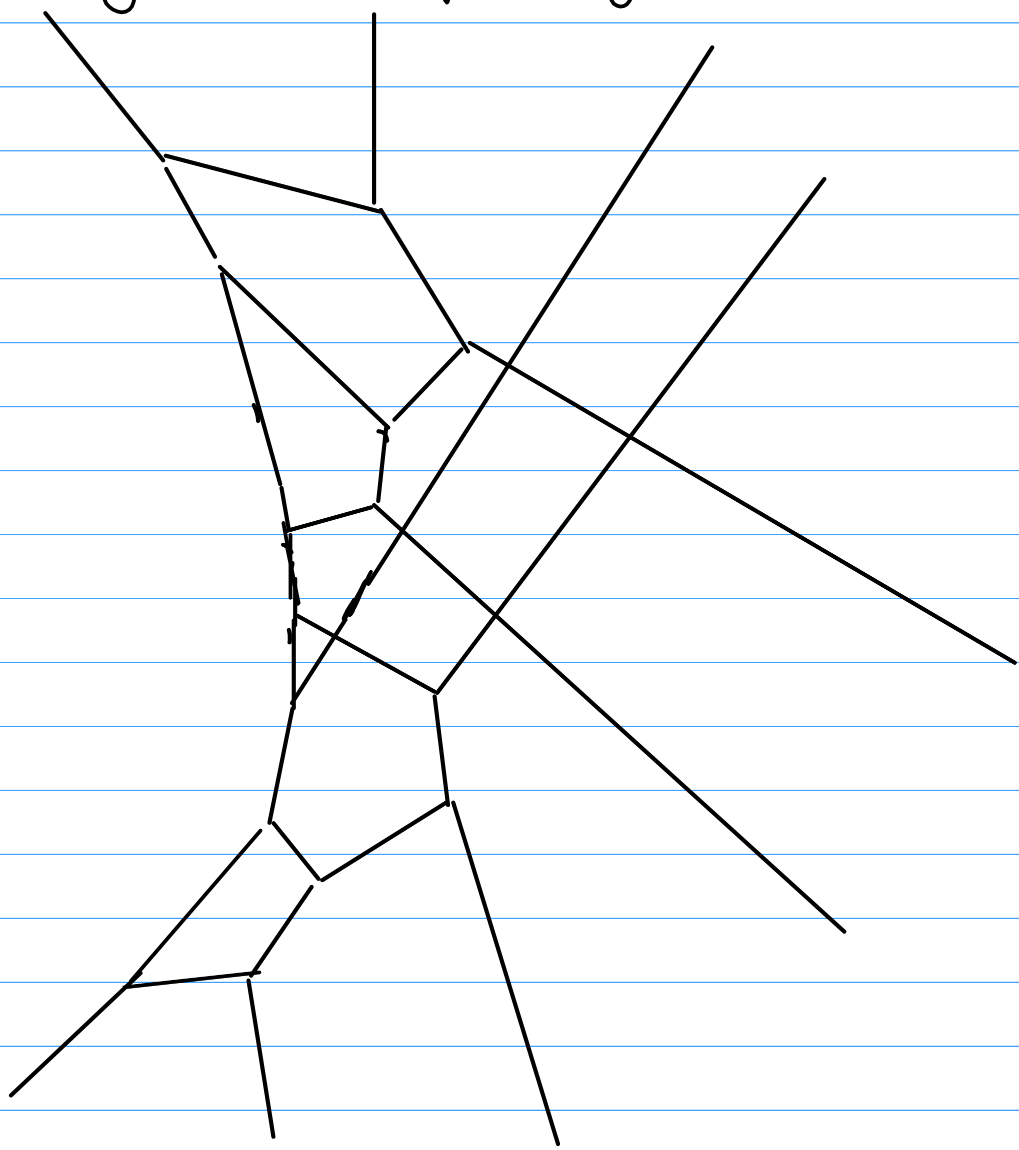
Q: Look at unbounded line-solitons. Observations?

Q: Remind you of plabic graphs?

Use the algorithm we saw in lecture to turn the previous contour plot into a "generalized plabic graph"



Ex: Consider soliton solution coming from point $A = (1, a, t) \in (Gr_{1,3})_{>0}$. Draw the contour plot $\mathcal{C}_t(u_A)$, & label the regions & edges.

It may be helpful to calculate how T_A behaves on the line $x = -cy$ (thinking of $c = \tan\varphi$ and φ the angle measured counterclock from pos. y-axis)