Introduction to The Dirichlet Space

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Zero Sets

What are the zero sets?

- What are the zero sets of functions in $\mathcal{D}$?
- Given $Z \subset \mathbb{D}$ $\exists ? f \in \mathcal{D} \setminus \{0\}$ $f|_Z = 0$.
- There is no complete description, I will describe some specific results.
- Perhaps the most noteworthy thing is the variety of tools used.

The Hardy Space Results

As background we recall the results for H^2

- Interior zero sets: $Z = \{z_i\} \subset \mathbb{D}$ is a zero set if and only if it satisfies the Blaschke condition $\sum (1 - |z_i|^2) < \infty$.
- Boundary zero sets: The boundary function $f(e^{i\theta})$ is, in general, only defined *a.e.*so some care must be taken in formulating the question. If E is a closed subset of the boundary and $|E| = 0$ then there is a function in the disk algebra, and hence in H^2, that vanishes precisely on E.

General Results

Consider the set $Z = \{z_i\} = \left\{r_n e^{i\theta_n}\right\} \subset \mathbb{D}$ which might satisfy

$$\sum (1 - r_i) < \infty. \tag{BI}$$

$$\sum \left|\log(1 - r_i)\right|^{-1+\varepsilon} < \infty. \tag{A_ε}$$

- Because $\mathcal{D} \subset H^2$ condition (BI) is necessary for Z to be a zero set.
- Carleson (1952): If (A_ε) holds for some $\varepsilon > 0$ then for every choice of $\{\theta_n\}$, Z is a zero set. For no $\varepsilon < 0$ does the condition (A_ε) suffice to insure that Z is a zero set for every choice of $\{\theta_n\}$.

- Shapiro-Shields (1962): If (A_ε) holds for $\varepsilon = 0$ then Z is a zero set for any choice of $\{\theta_n\}$. That is the best possible condition depending only on the $\{r_n\}$.
  - Proof discussion: Recall that $m_{z_i 0}(z)$ is the multiplier which is zero at z_i and maximal at the origin. Consider the product $P(z) = \prod_i m_{z_i 0}(z)$.
  - (If we solve the Hardy space version of the multiplier extremal problem used to define $m_{z_i 0}(z)$ we obtain an individual Blaschke factor. Thus $P(z)$ can be viewed as a "generalized Blaschke product".)
  - Because each individual factor has modulus at most one the product either converges to a holomorphic function with zeros at exactly $\{z_i\}$ or diverges to the function which is identically zero. Because the factors have multiplier norm one the product will be a multiplier and hence, in particular, in the Dirichlet space.
  - We test which case holds by evaluating at $z = 0$. We find that we have convergence if $P(0) = \prod \delta(0, z_i) > 0$, or, equivalently, if (A_ε) holds for $\varepsilon = 0$.
  - This is not an alternative to the SS proof, it is a recasting of their proof in convenient (for us) language.

- Nagel-Rudin-Shapiro (1982): If Z fails to satisfy (A_ε) for $\varepsilon = 0$ then there is a choice of $\{\theta_n\}$ for which $\left\{r_j e^{i\theta_j}\right\}$ is not a zero set.
- Proof discussion: Because the series diverges it is possible to chose the $\{\theta_n\}$ so that each approach region, $NRS(e^{i\theta})$, contains infinitely many of the $\{z_n\}$. The NRS theorem insures that, for *a.e.* θ, the boundary function $f(ei^{\theta})$ can be obtained by taking the limit through $NRS(e^{i\theta})$. Hence if f vanishes at all the $\{z_n\}$ then it must have $f(e^{i\theta}) = 0$ *a.e.* and hence must be the zero function.

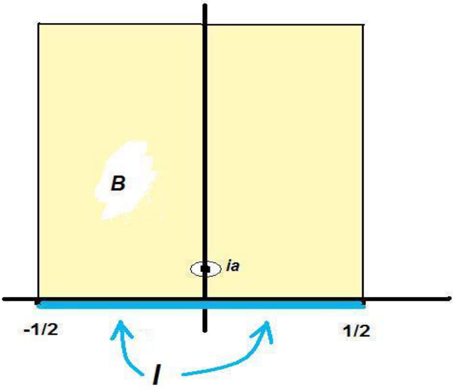
Sets With a Single Accumulation Point

Some effort has been spent trying to understand the, presumable easier, special case where Z only has one accumulation point;

$$\bar{Z} \cap \mathbb{T} = \{1\} \tag{SAP}$$

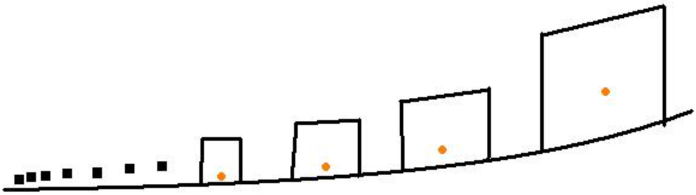
- If Z is in a single radius, say the positive real axis, (BI) is also sufficient. Proof: $B_Z(z)(1-z)^2 \in \mathcal{D}$.
- The same formula also covers the case of Z which satisfies (BI) and (SAP) and lies in a nontangential approach region.
- Caughran (1969): There is a Z which satisfies (BI) and (SAP) which is not a zero set

- Richter-Ross-Sundberg (2004): If Z fails to satisfy (A_ε) for $\varepsilon = 0$ then there is a choice of $\{\theta_n\}$ for which $Z = \{r_n e^{i\theta n}\}$ satisfies (SAP) and is not a zero set.
- Discussion: The proof is a "bare hands" classical function theory proof. RRS prove a Lemma which is a quantitative version of the fact that, for a holomorphic function f defined on B,



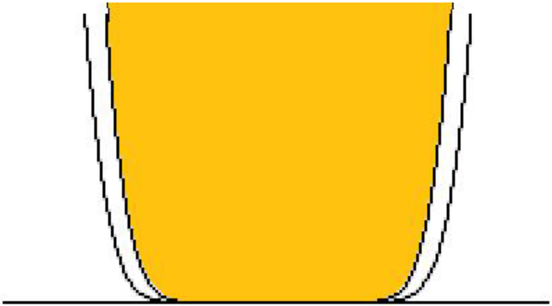
- these three statements can't all be true:

1. f has a zero near the boundary of B; $f(ia) = 0$ for some small $a > 0$,
2. f has limited oscillation on B; $\int_B |f'|^2$ is small, and
3. f stays away from 0 on the boundary of B; $-\int_I 0 \wedge \log |f|$ is small.



- If g has zeros as indicated in the picture, one in each box, then, by the Lemma, either 2. is violated infinitely often which forces $\mathcal{D}(g) = \infty$ and thus $g \notin \mathcal{D}$; or 3. is violated infinitely often which forces (log) to be violated and g to be identically zero.

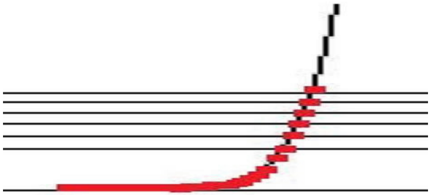
- Mashreghi and Shabankhah (2009): However, if Z satisfies (SAP) and stays inside a region quantitatively smaller than NRS(1) then Z is a zero set.



$$y = \exp\left(-1/\left|x\right|\right),\ y = \exp\left(-1/\left|x\right|^{.95}\right)$$

$$\text{(BI)} + \text{in yellow} \implies \text{zero set}$$

- Let's do this on the halfplane. Suppose $y_n = n^{-1-\beta}$ for some $\beta > 0$ and the zeros are located where the curve has height y_n



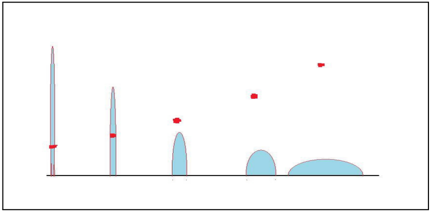
Location of Zeros

- (The general case is not much different from this example.) Thus

$$z_n = x_n + iy_n = \left(\frac{1}{(1+\beta)\log n}\right)^{1/.95} + i\frac{1}{n^{1+\beta}}.$$

- We want to know if we can find a function f in $\mathcal{D}$ with that zero set, Z. We would have $f = cB_f S_f O_f$. By the comments after Carleson's formula we see $O_f \in \mathcal{D}$. From that formula we also see that if f works then so does the modification with $cB_f S_f$ replaced by B_Z

- We are reduced to the following question: Z is given. Consider $d\nu_Z(\theta) = \sum P_{z_i}(e^{i\theta})d\theta$, an infinite positive measure which is locally finite except at $z = 1$. As suggested by the picture, there is not much overlap between the mass associated with different P_{z_i}.



The density for $d\nu_Z$

- We want to find an outer function $F \in \mathcal{D}$ so that

$$\int_T |F|^2 \, d\nu_Z(\theta) < \infty$$

As the picture suggests,

$$\int_T |F|^2 \, d\nu_Z(\theta) \sim \sum |F(x_n)|^2$$

- There is now a tension between two constraints. If we make $|F|^2$ very small everywhere near the origin then we are in danger of violating (log). On the other hand if we make $|F|^2$ small only on the primary support of ν_Z and, say, $|F|^2 = 1$ otherwise, then we will make $|F|$ very rough and perhaps generate a large derivative on the interior, taking us out of the Dirichlet space. Because the interior values of F are given by the formula (in the disk case)

$$F(z) = \exp\left\{ \frac{1}{2\pi} \int_{\mathbb{T}} \frac{e^{it} + z}{e^{it} - z} \log\left|F\left(e^{it}\right)\right| dt \right\},$$

  the interior oscillation of $F(z)$ is hard to analyze precisely; $|F'(z)|$ is related to $\left|F\left(e^{it}\right)\right|$ in a complicated nonlinear way. In fact there is no satisfactory systematic approach to showing $F \in \mathcal{D}$.
- If we are willing to make $|F|$ smooth then we can avoid the second problem; it is a theorem of Carleson and Jacobs [**?**] that if $\left|F\left(e^{it}\right)\right|$ is smooth then the outer function $F(z)$ will extend to be smooth on the closed disk, and hence will automatically be in $\mathcal{D}$. This approach costs us flexibility and almost certainly prevents us from getting an optimal result, however it does leave room for a positive result.

- Suppose we define F near the origin by $|F(x)|^2 = \exp(-1/|x|^{.95})$ and have it smooth and bounded elsewhere. We have

$$\begin{aligned}\int_T |F|^2 \, d\nu_Z(\theta) &\sim \sum |F(x_n)|^2 \\ &\sim \sum \exp\left(-1/|x_n|^{.95}\right) \\ &= \sum \exp\left(\left(\log \frac{1}{n^{1+\beta}}\right)^{.95}\right)^{1/.95} \\ &= \sum \frac{1}{n^{1+\beta}} < \infty.\end{aligned}$$

- Our other constraint is (log):

$$\int_0 \left|\log |F|^2\right| \sim \int_0 \frac{1}{|x|^{.95}} < \infty.$$

- We are OK!
- Trying to work with the NRS region rather than the yellow one would lead to trying to use the previous argument with .95 replaced by 1 in which case the argument fails.

Inside and Outside

- The Dirichlet space sits inside the Hardy space H^2 and contains the space A^∞ of holomorphic functions on the disk which extend to be C^∞ on the closed disk:

$$A^\infty \subset \mathcal{D} \subset H^2$$

- Ideas and results from both the containing space and the contained space are frequently used to study the Dirichlet space. We saw an example of each in the previous proof.
  - The Carleson-Jacobs theorem insured that the outer function we constructed was in A^∞ and hence in $\mathcal{D}$.
  - The constraint (log) for functions in H^2 showed that there was no easy way to replace the exponent .95 in our example by 1.

Boundary Zero Sets

- The situation is complicated and not well understood; and the methods are rather different than those I have been discussing. I will just mention a few results for flavor.
- $\mathcal{D} \subset H^2$ hence boundary zero sets must have measure zero.
- If E is a closed set of capacity zero then, by work of Brown and Cohn refining earlier work by Carleson, there is an $f \in \mathcal{D} \cap A(\mathbb{D})$ with zero set exactly E.
- Suppose E is a closed subset of the circle with complementary intervals $\{I_n\}$. The following is due to several people independently: If $\sum |I_n| = 2\pi$ (so $|E| = 0$) and $\sum |I_n| \, |\log |I_n|| < \infty$ (so E is a *Carleson set)* then $\exists f \in A^\infty \subset \mathcal{D}$ with zero set exactly E.