Intersection theory on toric hyperkähler varieties

In this talk I decided to sketch some ideas showing why toric hyperkähler varieties are interesting; instead of detailed and technical definitions and results. If you are interested in such approach I can warmly suggest our recent paper ~~on~~ math.AG/0203096 with Bernd Sturmfels. Here are some copies of this paper still warm.

I will sketch 3 aspects of intersection theory on toric hyperkähler varieties.

1, stringy cohomology $\rightsquigarrow$ mirror symmetry
2, cohomology ring $\rightsquigarrow$ Hard Lefschetz
3, L^2 cohomology $\rightsquigarrow$ intersection form

1, Stringy cohomology

$\mathcal{M}_G$ = hyperkähler moduli space of semistable Higgs G-bundles

Conjecture (Hausel, Thaddeus) (announcement in math.AG/0106140)

$$H^*_{st}\left(\mathcal{M}_{SL(n)}\right) = H^*_{st}\left(\mathcal{M}^k_{PGL(n)}\right)$$

has serious singularities

$(k,n)=1$ it is an orbifold

$$H^*_{st} = H^* + \text{stringy contribution from singularities}$$

generally speaking a hyperkähler singularity is modelled infinitesimally on $V\oplus V^*////G$ where $G \subset GL(V)$

$V\oplus V^*////G = \mu_G^{-1}(0)//G$, where $\mu_G : V\oplus V^* \to \mathfrak{g}^*$ is the moment map the stringy contributions are of the form:

$$H^*_{St}(V \oplus V^* /\!/\!/\!/ G) = ?$$

working hypothesis: it could be obtained from

$$H^*_{St}(\underbrace{V \oplus V^* /\!/\!/\!/ T}_{\text{(affine) toric hyperkähler variety}})^{W}$$

maximal torus; $W \leftarrow$ Weyl group.

2. Cohomology ring

Defn (Bielawski – Dancer, 2000; Hausel–Sturmfels, 2002)

Let $A : \mathbb{Z}^n \twoheadrightarrow \mathbb{Z}^d$ define $\mathbb{T}^d_{\mathbb{C}} \subset \mathbb{T}^n_{\mathbb{C}} \curvearrowright \mathbb{C}^n$
$\quad \| \; [a_1, \ldots, a_n]$

Then $\mu_{\mathbb{T}^d_{\mathbb{C}}} = \mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{C}^d$
$\quad (z, w) \mapsto \sum z_i \cdot w_i \cdot a_i$

$$Y(A, \theta) := \mu^{-1}_{\mathbb{T}^d_{\mathbb{C}}}(0) /\!/_{\theta} \mathbb{T}^d_{\mathbb{C}}, \text{ where } \theta \in \mathbb{Z}^d$$

now $Y(A,\theta) \to Y(A,0)$ crepant $\Rightarrow$

$$H^*_{St}(Y(A,0)) = H^*_{St}(Y(A,\theta)) = ?$$

for generic $\theta \in \mathbb{Z}^d$ it is an orbifold

in this talk answer $H^*(Y(A,\theta)) = ?$

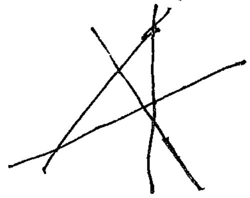
htpy type of $Y(A,\theta)$: $\mathbb{T}^{n-d}_{\mathbb{C}} \curvearrowright Y(A,\theta)$ with

$\mu_{\mathbb{T}^{n-d}_{\mathbb{C}}} : Y(A,\theta) \to \mathbb{C}^{n-d}$ moment map

$C^{ex} = \mu^{-1}_{\mathbb{T}^{n-d}_{\mathbb{C}}}(0) \sim Y$

"extended core" $\qquad \mathbb{T}^{n-d}_{\mathbb{R}} \curvearrowright C^{ex}$ with moment map

$\mu_{T^{n-d}_{\mathbb{R}}} : \mathbb{C}^{\alpha} \to \mathbb{R}^{n-d}$



$\mathcal{H}$ affine hyperplane arrangement
$F \in |\mathcal{H}|$ region

$\mu^{-1}_{T^{n-d}_{\mathbb{R}}}(F) =$ ~~[illegible]~~ toric orbifold. X_F

Thm (Konno 2000, Hausel-Sturmfels 2002)

$$H^*(Y, \mathbb{Q}) = \mathbb{Q}[x_1, \dots, x_n] \Big/ M(\mathcal{H}) + \underbrace{\mathrm{Cir}(A)}_{}$$

matroid ideal of the m. of $\mathcal{H}$ — certain linear relations from A

"2 line Proof" (H-S)

$$Y = \mu^{-1}_{T^d_{\mathbb{C}}}(0) /\!/_{\theta} T^d_{\mathbb{C}} \sim X = \mathbb{C}^{2n} /\!/_{\theta} T^d_{\mathbb{C}}$$

$\mu_{T^d_{\mathbb{C}}}$

$\downarrow$ ← top. trivial.

$\mathbb{C}^d$

X is a Lawrence toric variety its cohom. is what you expect.

Corollary $b_{2i}(Y) = h_i(\mathrm{mat}(\mathcal{H}))$

thm (injective Hard Lefschetz H-S)

$[\omega] \in H^2(Y)$ ~~Kähler~~ ample class

$$L : H^{2i-2}(Y) \to H^{2i}(Y)$$
$$\alpha \mapsto \alpha \cup [\omega]$$

is injective for $i \le \frac{n-d}{2}$

$\Rightarrow$ new numerical conditions on $h_i(\text{matroid}(\mathcal{H}))$

Problem: is there a Hard Lefschetz for hyperkähler manifolds?

For this

3, L^2 – cohomology

$$H^*_{L^2}(Y) := \text{space of } L^2 \text{ harmonic forms on } Y$$

<u>Thm</u> (Hitchin 2000)

1, $H^k_{L^2}(Y) = 0$ $k \neq \text{middle} = 2n-2d$

2, L^2 harmonic m-forms are all (anti) self-dual

<u>Corollary</u>: $\operatorname{Im}\left(H^{2n-2d}_{cpt}(Y) \to H^{2n-2d}(Y) \right)$

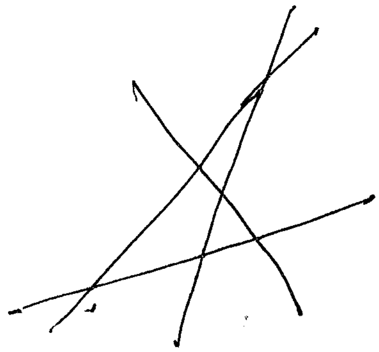
is (anti) self-dual

$\rightsquigarrow$ intersection form on $H^{2n-2d}_{cpt}(Y)$ is semi-definite.

<u>Conjecture</u>: 1, intersection form is definit
2, $H^{2n-2d}_{L^2}(Y) \cong \operatorname{Im}\left(H^{2n-2d}_{cpt}(Y) \to H^{2n-2d}_{cpt}(Y) \right)$
$\overset{1)}{\cong} H^{2n-2d}(Y)$

<u>remarks</u>: a, 1) is proven by Nakajima for quiver varieties.
b, in the quiver case 2, is conjectured by Vafa-Witten (1994)
c, 2, is proven for generic Y Hausel - Hunsicker - Mazzeo.

finally,
Combinatorial description of intersection form



$F \in |\mathcal{H}^{bd}|$ top $(n-d)$ dim. bounded region

$[X_F]$ generate $H^{2n-2d}_{cpt}(Y)$

$$\# \{ X_{F_i} \cap X_{F_j} \} = (-1)^{\dim F_i \cap F_j} \# \{ \mathrm{vert}(F_i \cap F_j) \}$$

definit?