Monotonicity of quantum relative entropy revisited

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Let D_1 and D_2 be density matrices on $\mathcal{H}$.

In this talk all the density matrices have strictly positive eigenvalues.

Relative entropy:

$$S(D_1, D_2) = \mathrm{Tr} D_1 (\log D_1 - \log D_2)$$

if

$$supp(D_1) \subset supp(D_2),$$

and $+\infty$, otherwise.

$$S(D_1, D_2) \leq \log n - \log \lambda,$$

where n is $\dim(\mathcal{H})$ and $\lambda > 0$ is the smallest eigenvalue of D_2.

Let $\mathcal{K}$ be another Hilbert space. Let $B(\mathcal{H})$ and $B(\mathcal{K})$ be matrix algebras.

The linear mapping $T : B(\mathcal{H}) \longrightarrow B(\mathcal{K})$ is coarse graining if T is trace preserving and 2-positive.

Theorem 1 (Uhlmann, 1977)

$$S(D_1, D_2) \geq S(T(D_1), T(D_2)).$$

$$\Delta a = D_2 a D_1^{-1} \quad (a \in B(\mathcal{H}))$$

is the relative modular operator.

$$\Delta = LR,$$

$$La = D_2 a, \text{ and } Ra = aD_1^{-1}.$$

Since

$$\log \Delta = \log L + \log R,$$

we have Araki's definition of relative entropy in a general von Neumann algebra

$$\begin{aligned} S(D_1, D_2) &= \left\langle D_1^{1/2}, (\log D_1 - \log D_2) D_1^{1/2} \right\rangle \\ &= -\left\langle D_1^{1/2}, (\log \Delta) D_1^{1/2} \right\rangle. \end{aligned}$$

Let D_1 and $T(D_1)$ be invertible matrices. Set

$$\Delta a = D_2 a D_1^{-1} \quad (a \in B(\mathcal{H})),$$

$$\Delta_0 x = T(D_2)\, x T(D_1)^{-1} \quad (x \in B(\mathcal{K})).$$

Then

$$\begin{aligned} S(D_1, D_2) &= -\left\langle D_1^{1/2}, (\log \Delta) D_1^{1/2} \right\rangle \\ &= \int_0^\infty \left\langle D_1^{1/2}, (\Delta + t)^{-1} D_1^{1/2} \right\rangle - (1+t)^{-1}\, dt. \end{aligned}$$

$$\begin{aligned} S(T(D_1), T(D_2)) &= -\left\langle T(D_1)^{1/2}, \log(\Delta_0)\, T(D_1)^{1/2} \right\rangle \\ &= \int_0^\infty \left\langle T(D_1)^{1/2}, (\Delta_0 + t)^{-1} T(D_1)^{1/2} \right\rangle \end{aligned}$$

$$-(1+t)^{-1}\,dt,$$

where

$$\log x = \int_0^\infty (1+t)^{-1} - (x+t)^{-1}\,dt.$$

This is enough to show that

$$\left\langle T(D_1)^{1/2}, (\Delta_0+t)^{-1} T(D_1)^{1/2} \right\rangle \leq \left\langle D_1^{1/2}, (\Delta+t)^{-1} D_1^{1/2} \right\rangle.$$

Set,

$$VxT(D_1)^{1/2} = T^*(x)\, D_1^{1/2}.$$

Then

$$\|V\| \leq 1, \quad V^*\Delta V \leq \Delta_0,$$

$$(\Delta_0+t)^{-1} \leq (V^*\Delta V+t)^{-1} \leq V^*(\Delta+t)^{-1} V.$$

Since

$$VT(D_1)^{1/2} = D_1^{1/2},$$

this implies

$$\left\langle D_1^{1/2}, (\Delta+t)^{-1} D_1^{1/2} \right\rangle \geq \left\langle T(D_1)^{1/2}, (\Delta_0+t)^{-1} T(D_1)^{1/2} \right\rangle.$$

The case of equality can be studied.

From operator inequality, differentiating by t,

$$V^*(\Delta+t)^{-1} D_1^{1/2} = (\Delta_0+t)^{-1} T(D_1)^{1/2}.$$

Then we obtain

$$V(\Delta_0 + t)^{-1} T(D_1) = (\Delta + t)^{-1} D_1^{1/2}.$$

By Stone-Weierstrass approximation,

$$V f(\Delta_0) T(D_1)^{1/2} = f(\Delta) D_1^{1/2}.$$

For $f(x) = x^{it}$ $(t \in \mathbb{R})$,

$$T^* \Big(\underbrace{T(D_2)^{it} T(D_1)^{-it}}_{u_t}\Big) = \underbrace{D_2^{it} D_1^{-it}}_{w_t},$$

which is necessary and sufficient condition for the equality.

Note that u_t and w_t are unitaries.

$$\mathcal{A}_T := \left\{ \begin{array}{c} X \in B(\mathcal{H}) : T(X^*, X) = T(X) T(X^*) \\ \text{and } T(X^* X) = T(X^*) T(X) \end{array} \right\}$$

is a *-subalgebra of $B(\mathcal{H})$ and

$$T(XY) = T(X) T(Y), \text{ for all } X \in \mathcal{A}_T \text{ and } Y \in B(\mathcal{H}).$$

The equality can be observed in a trivial way if there is S, such that

$$\begin{aligned} S(T(D_1)) &= D_1 \\ S(T(D_2)) &= D_2. \end{aligned}$$

(i) Double use of the monotonicity gives equality;

(ii) T^* algebraic automorphism "near D_1 and D_2".

This is the only case of equality:

$$\text{dual } [a, b]_D = \left\langle aD^{1/2}, \mathbf{1}b\mathbf{1}D^{1/2} \right\rangle,$$

(i) $[a, b]_D \geq 0$ if $a > 0$ and $b \geq 0$;

(ii) $[a, \mathbf{1}]_D =$TrDA

$$T^* = \alpha$$

$$\alpha(u_t) = w_t.$$

$$\alpha^{\#} \text{ dual of } \alpha \text{ w.r.t. } D_1 \text{ and } T(D_1).$$

$$\alpha^{\#} \cdot \alpha(u_t) = u_t,$$

$\alpha^{\#}\alpha$ leaves the states D_1 and D_2 invariants.

$$\text{Tr}\alpha^{\#}\alpha(a) D_i = \text{Tr}(aD_i).$$